Green's function for Poisson's Eqn.

$\nabla^2 u(\underline{x}) = F(\underline{x})$ with possibly non-homogeneous boundary conditions

Define $\nabla^2 G(\underline{x}; \underline{x}_0) = \delta(\underline{x} - \underline{x}_0)$

Note that the Green's function is the electrostatic potential for a point charge at $\underline{x} = \underline{x}_0$.

$$\int_{V} G(\underline{x};\underline{x}_0)\left\{ \nabla^2 u(\underline{x}) - F(\underline{x}) = 0 \right\} d\underline{x}$$

integrate by parts twice:

$$\int_{V} u(\underline{x})\, \nabla^2 G(\underline{x};\underline{x}_0)\, d\underline{x} = \int_{V} G(\underline{x};\underline{x}_0) F(\underline{x})\, d\underline{x}$$

$$- \int_{A} \left[G(\underline{x};\underline{x}_0) \underline{\nabla} u(\underline{x}) - u(\underline{x}) \underline{\nabla} G(\underline{x};\underline{x}_0) \right] \cdot \hat{n}\, dA$$

with $\nabla^2 G(\underline{x};\underline{x}_0) = \delta(\underline{x}-\underline{x}_0)$ we

find an explicit formula for $u(\underline{x}_0)$:

$$u(\underline{x}_0) = u_p(\underline{x}_0) + u_h(\underline{x}_0)$$

or if we interchange the labels $\underline{x}$, $\underline{x}_0$

$\Rightarrow$

$$u(\underline{x}) = \int_{V_0} G(\underline{x}_0 ; \underline{x}) \, F(\underline{x}_0) \, dV_0$$

$$- \int_{A_0} \left[G(\underline{x}_0 ; \underline{x}) \nabla_0 u(\underline{x}_0) - u(\underline{x}_0) \nabla_0 G(\underline{x}_0 ; \underline{x}) \right] \cdot \hat{n}_0 \, dA_0$$

Be careful! These are derivatives wrt $\underline{x}_0$; $\hat{n}_0$, dA_0, dV_0, etc.

* Need to solve $\nabla^2 G(\underline{x} ; \underline{x}_0) = \delta(\underline{x} - \underline{x}_0)$

2nd [helpful if $G(\underline{x} ; \underline{x}_0) = G(\underline{x}_0 ; \underline{x})$ symmetric]

* Need to choose appropriate b.c.s

1st For $G(\underline{x} ; \underline{x}_0)$

As for the ODE case, choose b.c.s for $G(\underline{x} ; \underline{x}_0)$ such that the RHS above is completely known

Simplest cases

① If $u(\underline{x})$ is given on Ω ("Dirichlet")

$\Rightarrow$ Choose $G(\underline{x}\,;\,\underline{x}_0) = 0$ on Ω

② If $\underline{\nabla}\, u(\underline{x}) \cdot \hat{n}$ is given on Ω

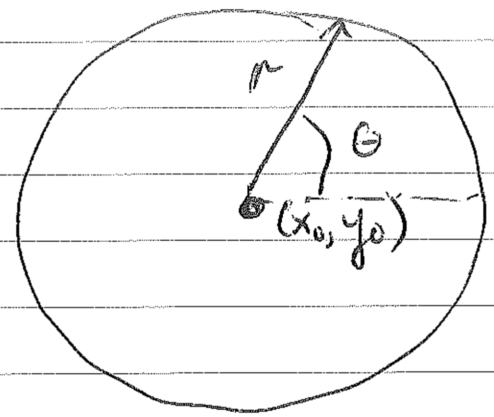
("Newmann") $\Rightarrow$ choose $\underline{\nabla} G(\underline{x}\,;\,\underline{x}_0) \cdot \hat{n} = 0$ on Ω

[In 2D, the volume integrals become area integrals and the boundary terms are closed loop line integrals]

Lets compute the infinite space Green's function in 2D corresponding to the electrostatic potential of a single point charge in a plane (no bounderies)

2D

$$\nabla^2 G(\underline{x};\underline{x}_0) = \delta(\underline{x}-\underline{x}_0)$$



$$\underline{x} = (x, y)$$
$$\underline{x}_0 = (x_0, y_0)$$

Physically: $G(\underline{x}, \underline{x}_0)$ depends on the distance from the charge / source but not the angle

$$\Rightarrow \quad G(\underline{x};\underline{x}_0) = G(r,\theta) = G(r)$$

where $r = |\underline{r}| = |\underline{x}-\underline{x}_0| = \sqrt{(x-x_0)^2+(y-y_0)^2}$

$$\underline{x}-\underline{x}_0 = re^{i\theta}$$

$$x-x_0 = r\cos\theta \quad ; \quad y-y_0 = r\sin\theta$$

2 steps to find $G(\underline{x};\underline{x}_0)$

* Solve $\nabla^2 G(\underline{x};\underline{x}_0) = 0 \qquad \underline{x} \neq \underline{x}_0$

* Integrate <u>around</u> the singularity at $\underline{x}=\underline{x}_0$

[like integrating <u>across</u> the singularity in 1D]

$\boxed{\text{Step 1}}$ No θ-dependence $\Rightarrow$

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dG(r)}{dr}\right) = 0 \;;\quad r\frac{dG(r)}{dr} = C$$

$$\Rightarrow \boxed{G(r) = C\ln r + D}$$

$\boxed{\text{Step 2}}$ $$\int_A \nabla^2 G(\underline{x};\underline{x}_0)\,dA = \int_A \delta(\underline{x}-\underline{x}_0)\,dA = \underline{1}$$

$\{x_0 \text{ in } A\}$ By divergence thm

$$= \oint_S \underline{\nabla} G(\underline{x};\underline{x}_0)\cdot\hat{n}\,dS$$

with $\underline{\nabla} = \frac{\partial}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial}{\partial\theta}\hat{\theta}$

Since $G = G(r)$ we find $\{\text{consider a circle}\}$

$$= \int_0^{2\pi} \frac{dG(r)}{dr}\,\hat{r}\cdot\hat{r}\;r\,d\theta = \underline{1} \quad \{r \text{ fixed}\}$$

$\underline{\nabla} G$ is in the $\hat{r}$-direction ; $\hat{n}$ is in the $\hat{r}$-direction

$$\Rightarrow \quad 2\pi\, r \frac{d\,G(r)}{dr} = 1$$

or $\quad \dfrac{dG(r)}{dr} = \dfrac{1}{2\pi r}$ as we had before with $C = \dfrac{1}{2\pi}$

Thus $G(\underline{x};\underline{x}_0) = \dfrac{\ln r}{2\pi} + D$ is the infinite space Green's Function in 2D, determined up to an arbitrary additive constant

[like the electrostatic potential is determined up to a constant]

How can we use this to solve problems in

$\boxed{\text{Bounded Domains}}$?

Method of Images ...

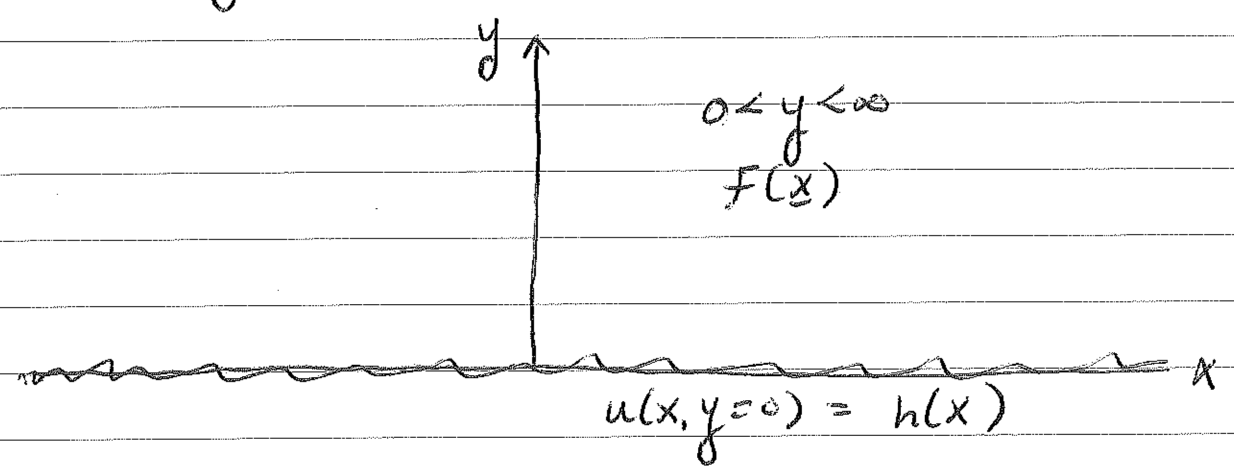
Check symmetry: $G(\underline{x};\underline{x}_0) = G(\underline{x}_0;\underline{x}) =$

$$\frac{1}{2\pi} \ln \sqrt{(x-x_0)^2 + (y-y_0)^2}$$

Method of Images for a semi-infinite domain problem in 2D

$$\nabla^2 u(\underline{x}) = f(\underline{x}) \qquad u(x, y=0) = h(x)$$

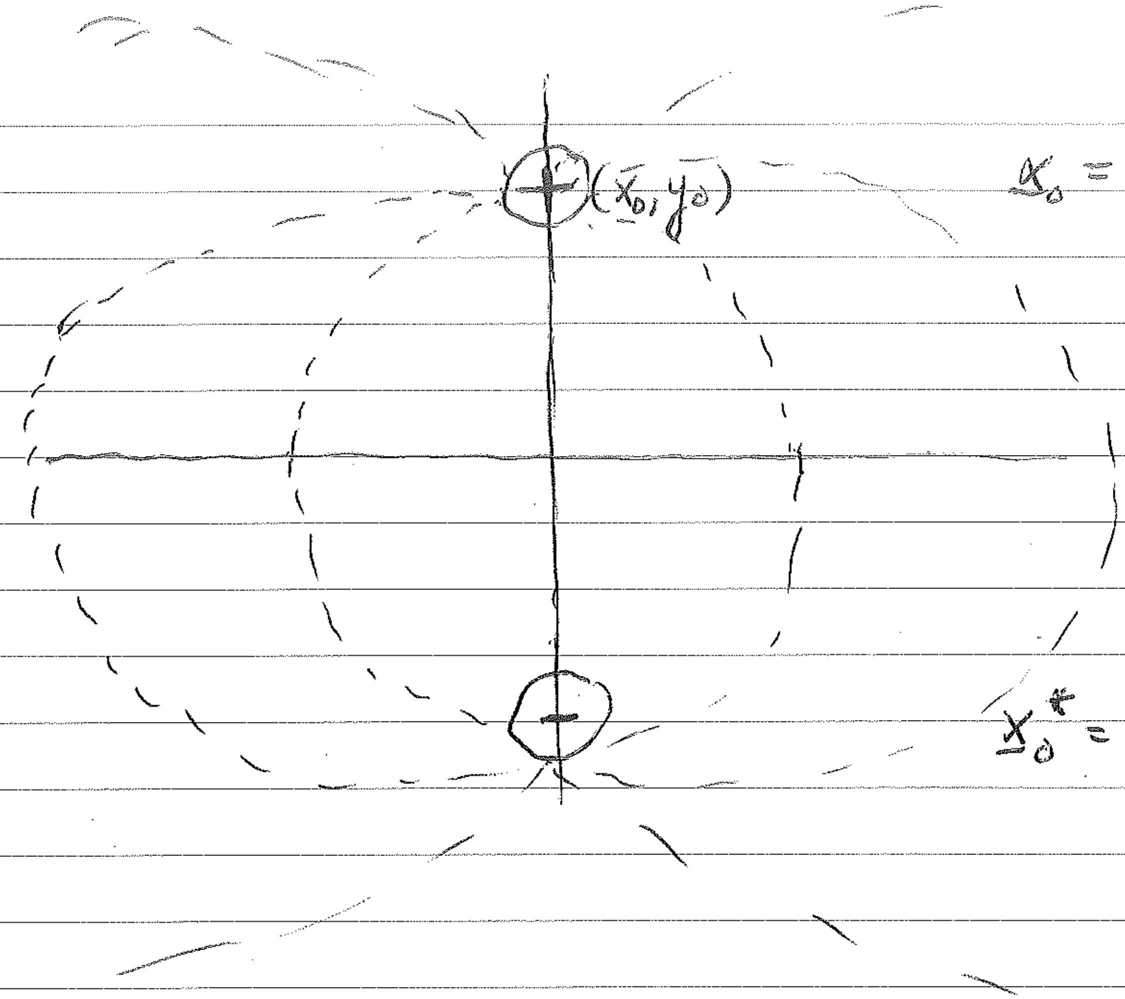
$$0 < y < \infty \;, \quad -\infty < x < \infty$$



We want to solve $\nabla^2 G(\underline{x}\,;\underline{x}_0) = \delta(\underline{x}-\underline{x}_0)$

with $G(x, 0\,; x_0, y_0) = 0$ {the homogeneous version of the b.c.s for $u(\underline{x})$}.

To satisfy the homogeneous b.c.s for G :
introduce an image charge



$\underline{x}_0 = (x_0, y_0)$

$\underline{x}_0^* = (x_0, -y_0)$

In infinite space $\nabla^2 G(\underline{x}; \underline{x}_0) = \delta(\underline{x} - \underline{x}_0) - \delta(\underline{x} - \underline{x}_0^*)$

in $-\infty < x < \infty$, $-\infty < y < \infty$

which still satisfies $\nabla^2 G(\underline{x}; \underline{x}_0) = \delta(\underline{x} - \underline{x}_0)$
in the upper half plane

The problem is linear $\Longrightarrow$

$$G(\underline{x}; \underline{x}_0) = \frac{1}{2\pi} \ln|\underline{x} - \underline{x}_0| - \frac{1}{2\pi} \ln|\underline{x} - \underline{x}_0^*|$$

$$= \frac{1}{2\pi} \ln\left\{\frac{|\underline{x}-\underline{x}_0|}{|\underline{x}-\underline{x}_0^*|}\right\}$$

$$= \frac{1}{2\pi} \ln\left\{\frac{\left[(x-x_0)^2+(y-y_0)^2\right]^{1/2}}{\left[(x-x_0)^2+(y+y_0)^2\right]^{1/2}}\right\}$$

Then using $\ln x^r = r \ln x \implies$

$$G(\underline{x};\underline{x}_0) = \frac{1}{4\pi} \ln\left\{\frac{(x-x_0)^2+(y-y_0)^2}{(x-x_0)^2+(y+y_0)^2}\right\}$$

Is it true that $G(\underline{x};\underline{x}_0)\Big|_{y=0} = 0$?

$$G(\underline{x};\underline{x}_0)\Big|_{y=0} = \frac{1}{4\pi} \ln\left\{\frac{(x-x_0)^2+y_0^2}{(x-x_0)^2+y_0^2}\right\}$$

$$= \frac{1}{4\pi} \ln 1 = 0 \quad \checkmark$$

Furthermore, $\nabla^2 G(\underline{x};\underline{x}_0) = \delta(\underline{x}-\underline{x}_0)$ in the upper half plane

Now the Formula: $u(\underline{x}) = \int_{A_0} G(\underline{x}_0; \underline{x}) F(\underline{x}_0)\, dA_0$

$$- \left[\oint_{S_0} G(\underline{x}_0; \underline{x})\, \underline{\nabla}_0 u(\underline{x}_0) \cdot \hat{n}_0\, dS_0 - \oint_{S_0} u(\underline{x}_0)\, \underline{\nabla}_0 G(\underline{x}_0; \underline{x}) \cdot \hat{n}_0\, dS_0 \right]$$

$$\int_{A_0} dA_0 = \int_{-\infty}^{\infty} dx_0 \int_0^{\infty} dy_0$$

$$\oint_{S_0} dS_0 = \int_{-\infty}^{\infty} dx_0 \quad \text{with} \quad y_0 = 0$$

$$\hat{n}_0 = -\hat{y}_0$$

$$\underline{\nabla}_0 G(\underline{x}_0; \underline{x}) \cdot \hat{n}_0 = \frac{\partial G}{\partial y_0}(\underline{x}_0; \underline{x})\,(-1)$$

boundary term is

$$\int_{-\infty}^{\infty} dx_0\, h(x_0) \frac{\partial G}{\partial y_0}(\underline{x}_0; \underline{x}) \Big|_{y_0=0} \quad (-1)$$

Now plug in:

$$G(x_0, y_0; x, y) = \frac{1}{4\pi} \ln\{(x_0-x)^2 + (y_0-y)^2\}$$

$$- \frac{1}{4\pi} \ln\{(x_0-x)^2 + (y_0+y)^2\}$$

$$\frac{\partial G}{\partial y_0} = \frac{1}{4\pi} \frac{2(y_0-y)}{(x_0-x)^2 + (y_0-y)^2}$$

$$- \frac{1}{4\pi} \frac{2(y_0+y)}{(x_0-x)^2 + (y_0+y)^2}$$

$$\left.\frac{\partial G}{\partial y_0}\right|_{y_0=0} = - \frac{2y}{4\pi} \frac{1}{(x_0-x)^2+y^2}$$

$$- \frac{2y}{4\pi} \frac{1}{(x_0-x)^2+y^2}$$

$$= - \frac{1}{\pi} \frac{y}{[(x_0-x)^2+y^2]}$$

$$u(\underline{x}) = \int_{-\infty}^{\infty} dx_0 \int_{0}^{\infty} dy_0 \quad G(\underline{x}_0 ; \underline{x}) \, f(\underline{x}_0)$$

$$+ \int_{-\infty}^{\infty} h(x_0) \, \frac{1}{\pi} \, \frac{y}{\left[(x_0-x)^2 + y^2\right]} \, dx_0$$