Green's Function for Poisson's Eqn.

$\nabla^2 u(x) = f(x)$ with possibly non-homogeneous boundary conditions.

Define $\nabla^2 G(x; x_0) = \delta(x - x_0)$.

Note that the Green's function is the electrostatic potential for a point charge at $x = x_0$. 
\[ \int_Y G(x; x_0) \nabla^2 u(x) - f(x) = 0 \int_Y \, dx \]

Integrate by parts twice:

\[ \int_Y u(x) \nabla^2 G(x; x_0) \, dx = \int_Y G(x; x_0) f(x) \, dx \]

\[ -\int_A \left[ G(x; x_0) \nabla u(x) - u(x) \nabla G(x; x_0) \right] \cdot n \, dA \]

With \( \nabla^2 G(x; x_0) = \delta(x - x_0) \) we find an explicit formula for \( u(x_0) \):

\[ u(x_0) = u_p(x_0) + u_n(x_0) \]

or if we interchange the labels \( x, x_0 \):

\[ \Rightarrow \]
\[ u(x) = \int_{V_0} G(x_0; x) f(x_0) \, dV_0 \]

\[ -\int_{A_0} \left[ \left( \frac{\partial G(x_0; x)}{\partial x} \right) \frac{\partial u(x_0)}{\partial x} + u(x_0) \frac{\partial G(x_0; x)}{\partial x} \right] n_0 \, dA_0 \]

**Be careful!** These are derivatives wrt \( x_0 \), \( n_0 \), \( dA_0 \), \( dV_0 \), etc.

*Need to solve \( \nabla^2 G(x_0; x) = \delta(x - x_0) \)

\[ \text{[helpful if } G(x_0; x) = G(x_0; x) \text{ symmetric]} \]

* Need to choose appropriate basis

\[ \text{For } G(x, x_0) \]

As for the ODE case, choose basis for \( G(x_0; x) \) such that the RHS above is completely known
Simplest cases

1. If $u(x)$ is given on $\Omega$ ("Dirichlet")

   $\Rightarrow$ choose $G(x, x_0) = 0$ on $\Omega$

2. If $\nabla u(x) \cdot \hat{n}$ is given on $\Omega$

   ("Neumann") $\Rightarrow$ choose $\nabla G(x, x_0) \cdot \hat{n} = 0$

   on $\Omega$

In 2D, the volume integrals become zero.
Integrals, and the boundary terms are closed loop line integrals.

Let's compute the infinite space Green's function in 2D corresponding to the electrostatic potential of a single point charge in a plane (no boundaries).
\[ V^2 G(x, x_0) = \delta(x - x_0) \]

\[ x = (x, y) \]
\[ x_0 = (x_0, y_0) \]

Physically, \( G(x, x_0) \) depends on the distance from the charge/source but not the angle.

\[ \Rightarrow G(x, x_0) = G(r, \theta) = G(r) \]

where \( r = |x| = |x - x_0| = \sqrt{(x - x_0)^2 + (y - y_0)^2} \)

\[ x - x_0 = r e^{i\theta} \]

\[ x-x_0 = r \cos \theta \quad ; \quad y-y_0 = r \sin \theta \]

2 steps to find \( G(x, x_0) \)

1. Solve \( V^2 G(x, x_0) = 0 \) \( \forall x \neq x_0 \)
2. Integrate around the singularity at \( x = x_0 \)

\[ \text{[like integrating across the singularity in 1D]} \]
Step 1

No $\theta$-dependence \Rightarrow

\[ \frac{1}{r} \frac{d}{dr} \left( r \frac{dG(r)}{dr} \right) = 0 \quad \Rightarrow \quad r \frac{dG(r)}{dr} = C \]

\[ \Rightarrow \quad G(r) = C \ln r + D \]

Step 2

\[ \int_A \nabla^2 G(x; x_0) \, dA = \int_A G(x-x_0) \, dA = 1 \]

\( \exists x_0 \text{ in } A \)

By divergence theorem

\[ = \oint_S \nabla G(x; x_0) \cdot \hat{n} \, dS \]

with \( \nabla = \frac{1}{r} \frac{d}{dr} \hat{r} + \frac{1}{r \sin \theta} \frac{d}{d\theta} \hat{\theta} + \frac{1}{r \sin \theta} \frac{d}{d\phi} \hat{\phi} \)

Since \( G = G(r) \) we find

\[ = \oint_0^{2\pi} \frac{dG(r)}{dr} \hat{r} \cdot \hat{r} \, r \, d\theta = 1 \oint_0^{2\pi} \frac{dG(r)}{dr} \, r \, d\theta \]

\( \nabla G \) is in the \( \hat{r} \)-direction, \( \hat{n} \) is in the \( \hat{r} \)-direction.
\[ 2\pi r \frac{d6(r)}{dr} = 1 \]

or \[ \frac{d6(r)}{dr} = \frac{1}{2\pi r} \] as we had before with \[ C = \frac{1}{2\pi} \]

Thus \[ G(x, x_0) = \frac{\ln r}{2\pi} + D \text{ is the} \]

infinite space Green's function in 2D determined up to an arbitrary additive constant

[like the electrostatic potential is determined up to a constant]

How can we use this to solve problems in

**Bounded Domains**?

**Method of Images**

Check symmetry: \[ G(x, x_0) = G(x_0, x) = \frac{1}{2\pi} \ln \sqrt{(x-x_0)^2 + (y-y_0)^2} \]
Method of Images for a semi-infinite domain

\[ \nabla^2 u(x) = f(x) \quad u(x, y=0) = h(x) \]

\[ 0 < y < \infty, \quad -\infty < x < \infty \]

We want to solve \( \nabla^2 G(x; x_0) = f(x - x_0) \)

with \( G(x, 0; x_0, y_0) = 0 \) the homogeneous version of the b.c.s for \( u(x) \).

To satisfy the homogeneous b.c.s for \( u \), introduce an image charge.
In infinite space \( \nabla^2 G(x, x_0) = \delta(x - x_0) - \delta(x - x_0^*) \)

in \( -\infty < x < \infty, \quad -\infty < y < \infty \)

which still satisfies \( \nabla^2 G(x, x_0) = \delta(x - x_0) \)

in the upper half plane

The problem is linear \( \Rightarrow \)

\[
G(x, x_0) = \frac{1}{2\pi} \ln |x - x_0| - \frac{1}{2\pi} \ln |x - x_0^*| 
\]
\[ \frac{1}{2\pi} \ln \left\{ \frac{|x-x_0|^2}{|x-x_0|^2 + (y+y_0)^2} \right\} \]

\[ = \frac{1}{2\pi} \ln \left\{ \frac{(x-x_0)^2 + (y-y_0)^2}{(x-x_0)^2 + (y+y_0)^2} \right\}^{1/2} \]

Then using \( \ln x = n \ln x \Rightarrow \)

\[ G(x, x_0) = \frac{1}{4\pi} \ln \left\{ \frac{(x-x_0)^2 + (y-y_0)^2}{(x-x_0)^2 + (y+y_0)^2} \right\} \]

Is it true that \( G(x, x_0) \bigg|_{y=0} = 0 \)?

\[ G(x, x_0) \bigg|_{y=0} = \frac{1}{4\pi} \ln \left\{ \frac{(x-x_0)^2 + y_0^2}{(x-x_0)^2 + y_0^2} \right\} \]

\[ = \frac{1}{4\pi} \ln 1 = 0 \checkmark \]

Furthermore, \( V^2 G(x, x_0) = 0(x-x_0) \) in the upper half plane.
Now the formula is:
\[ u(x) = \int_{A_0} g(x_0, x) f(x_0) \, dA_0 \]

\[ - \left[ \int_{S_0} \nabla_v G(x_0, \hat{x}) \cdot \hat{n}_0 \, dS_0 \right. \]
\[ - \left. \int_{S_0} u(x_0) \nabla_v G(x_0, \hat{x}) \cdot \hat{n}_0 \, dS_0 \right] \]

\[ \int_{A_0} \, dA_0 = \int_{-\infty}^{\infty} dx_0 \int_{-\infty}^{\infty} dy_0 \]

\[ \int_{S_0} \, dS_0 = \int_{-\infty}^{\infty} dx_0 \quad \text{with} \quad y_0 = 0 \]

\[ \hat{n}_0 = -\hat{y}_0 \]

\[ \nabla_v G(x_0, \hat{x}) \cdot \hat{n}_0 = \frac{\partial}{\partial y_0} (x_0, \hat{x}) (-1) \]

The boundary term is:
\[ \int_{-\infty}^{\infty} \, dx_0 \ h(x_0) \left. \frac{\partial}{\partial y_0} (x_0, \hat{x}) \right|_{y_0 = 0} (-1) \]
Now plug in:

\[ C(x_0, y_0; x, y) = \frac{1}{4\pi^2} \ln \frac{(x_0-x)^2 + (y_0-y)^2}{(x_0-x)^2 + (y_0+y)^2} \]

\[
\frac{\partial C}{\partial y_0} = \frac{1}{4\pi} \frac{2(y_0-y)}{(x_0-x)^2 + (y_0-y)^2} \]

\[
= \frac{1}{4\pi} \frac{2(y_0+y)}{(x_0-x)^2 + (y_0+y)^2} \]

\[
= -\frac{2y}{4\pi} \frac{1}{(x_0-x)^2 + y^2} \]

\[
= -\frac{2y}{4\pi} \frac{1}{(x_0-x)^2 + y^2} \]

\[
= -\frac{1}{\pi} \frac{y}{[(x_0-x)^2 + y^2]^2} \]
\[ u(x) = \int_{-\infty}^{\infty} dx_0 \int_{-\infty}^{\infty} dy_0 \, G(x_0 \mid x \mid) f(x_0) \]

\[ + \int_{-\infty}^{\infty} h(x_0) \frac{1}{\pi} \left[ \frac{y}{(x_0-x)^2 + y^2} \right] dx_0 \]