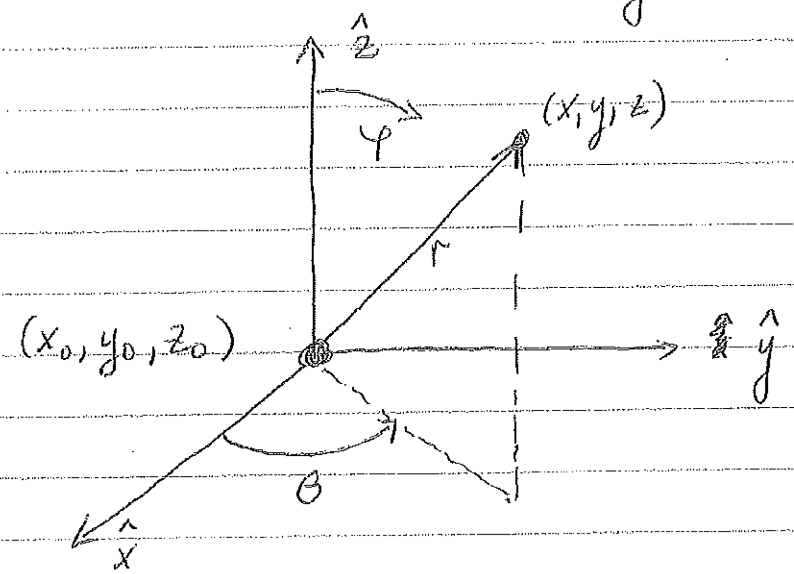
Math 322 The infinite space Green's function for use in the Method of Images:



We need to solve $\nabla^2 G(\underline{x}; \underline{x}_0) = \delta(\underline{x} - \underline{x}_0)$

and the idea is that the effect of this "source" at other values of $\underline{x}$ should depend only on r, but not on θ, φ, where

$$r = \left[(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 \right]^{1/2}$$

$$\Longrightarrow \nabla^2 G(\underline{x}; \underline{x}_0) = \nabla^2 G(r)$$

with ∇^2 in spherical coordinates given by

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{1}{\sin^2\varphi}\frac{\partial^2}{\partial\theta^2}$$

$$+ \frac{1}{r^2}\frac{1}{\sin\varphi}\frac{\partial}{\partial\varphi}\left(\sin\varphi\frac{\partial}{\partial\varphi}\right)$$

$$\Longrightarrow \quad \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dG(r)}{dr}\right) = 0 \qquad \underline{x} \neq \underline{x}_0$$

$$\Longrightarrow \quad r^2\frac{dG}{dr} = C \quad \Longrightarrow \quad G(r) = -\frac{C}{r} + D$$

Now lets integrate around the "source":

$$\int_{\forall} \nabla^2 G \, d\forall = \int_{\forall} \delta(\underline{x}-\underline{x}_0)\, d\forall = 1$$

$$\int_{\forall} \underline{\nabla}\cdot\underline{\nabla} G \, d\forall = \int_A \underline{\nabla} G \cdot \hat{n} \, dA$$

and lets consider a sphere at fixed r:

$\underline{\nabla} G$ is in the $\hat{r}$-direction

$\hat{n}$ is in the $\hat{r}$-direction

$dA = r^2 \sin\varphi \, d\varphi \, d\theta$ at fixed r

$$\int_0^{2\pi} d\theta \int_0^{\pi} \sin\varphi \, d\varphi \quad r^2 \frac{dG(r)}{dr} = 1$$

$$= r^2 \frac{dG}{dr} \left[-\cos\varphi \Big|_0^{\pi} \right] \theta \Big|_0^{2\pi} = 1$$

$$= 2 \cdot 2\pi \, r^2 \frac{dG}{dr} = 1 \quad \text{or} \quad \frac{dG}{dr} = \frac{1}{4\pi r^2}$$

and the constant $C = \frac{1}{4\pi} \implies$

$$\boxed{G(r) = -\frac{1}{4\pi r} + D}$$ determined up to an arbitrary constant D

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Green's function for other linear operators.

Consider the 1D Heat Eqn.

$$\frac{\partial u}{\partial t} = K\frac{\partial^2 u}{\partial x^2} + Q(x,t) \qquad \begin{array}{c} 0 < x < L \\ t > 0 \end{array}$$

$$u(x,0) = f(x)$$

$u(0,t)$, $u(L,t)$ given

or $\frac{\partial u}{\partial x}(0,t)$ $\frac{\partial u}{\partial x}(L,t)$ given

or mixed boundary conditions

Multiply by a Green's function and integrate "over the whole domain" which now includes time

$$\int_0^L dx \int_0^\infty dt \; G(x,t;x_0,t_0)\left[\frac{\partial u}{\partial t}(x,t) = K\frac{\partial^2 u(x,t)}{\partial x^2} + Q(x,t)\right]$$

Now integrate by parts:

(i) $$\int_0^\infty G \frac{\partial u}{\partial t}\, dt = -\int_0^\infty u \frac{\partial G}{\partial t}\, dt + uG\Big|_{t=0}^{\infty}$$

(ii) $$\int_0^L G \frac{\partial^2 u}{\partial x^2}\, dx = \int_0^L u \frac{\partial^2 G}{\partial x^2}\, dx$$

$$- \left[u \frac{\partial G}{\partial x} - G \frac{\partial u}{\partial x} \right]\Bigg|_{x=0}^{L}$$

So we arrive at

$$\int_0^L \int_0^\infty \left(-u \frac{\partial G}{\partial t} - k u \frac{\partial^2 G}{\partial x^2} \right) dx\, dt$$

$$= \int_0^L \int_0^\infty G Q \, dx\, dt - \int_0^L dx \;\; uG\Big|_{t=0}^{\infty}$$

$$-k \int_0^\infty dt \left[u \frac{\partial G}{\partial x} - G \frac{\partial u}{\partial x} \right]\Bigg|_{x=0}^{L}$$

Now choose $-\frac{\partial G}{\partial t} - k\frac{\partial^2 G}{\partial x^2} = \delta(x-x_0)\delta(t-t_0)$

$G = G(x,t;x_0,t_0)$

so that the LHS $= u(x_0,t_0)$

where $0 < x_0 < L$, $0 < t_0 < \infty$

Note that $G(x,t;x_0,t_0)$ satisfies a backwards heat equation

$$-\frac{\partial G}{\partial t} = k\frac{\partial^2 G}{\partial x^2} + \delta(x-x_0)\delta(t-t_0)$$

So time is in the opposite sense and the analogy to the homogeneous condition in time is

$$G(x,t;x_0,t_0) = 0 \quad \text{for} \quad t > t_0$$

Choose boundary conditions as usual

Then what happens to the formula:

$$u(x_0, t_0) = \int_0^L dx \int_0^\infty G(x,t)\, Q(x,t)$$ non-homogeneous term in the eqn.

$$+ \int_0^L dx \; F(x)\, G(x,t) \Big|_{t=0}$$ ← effects of the initial condition

$$- k \int_0^\infty dt \left[u \frac{\partial G}{\partial x} - G \frac{\partial u}{\partial x} \right] \Bigg|_{x=0}^{x=L}$$ ← boundary conditions

So we need to solve

$$-\frac{\partial G}{\partial t} = k\frac{\partial^2 G}{\partial x^2} + \delta(x-x_0)\delta(t-t_0) \qquad 0<t<t_0$$
$$0<x<L$$

* homogeneous boundary conditions in space

* $G=0 \quad t>t_0$

If we want to use the method of images with the infinite space Green's Function

$$-\frac{\partial G}{\partial t} = k\frac{\partial^2 G}{\partial x^2} + \delta(x-x_0)\delta(t-t_0) \qquad 0<t<t_0$$
$$-\infty<x<\infty$$

* $G(\pm\infty, t) = 0$

* $G(x,t) = 0 \quad t>t_0$

$$G(x,t) = \frac{1}{\sqrt{4\pi k(t_0-t)}}\exp\left[\frac{-(x-x_0)^2}{4k(t_0-t)}\right]$$

$$0<t<t_0$$

Example

$u_t = K u_{xx} + Q \,, \quad 0 \le x \le L, \quad 0 < t < \infty$

$u(0,t) = u(L,t) = 0 \qquad u(x,0) = f(x)$

Use the method of images with infinite-space Green's function:

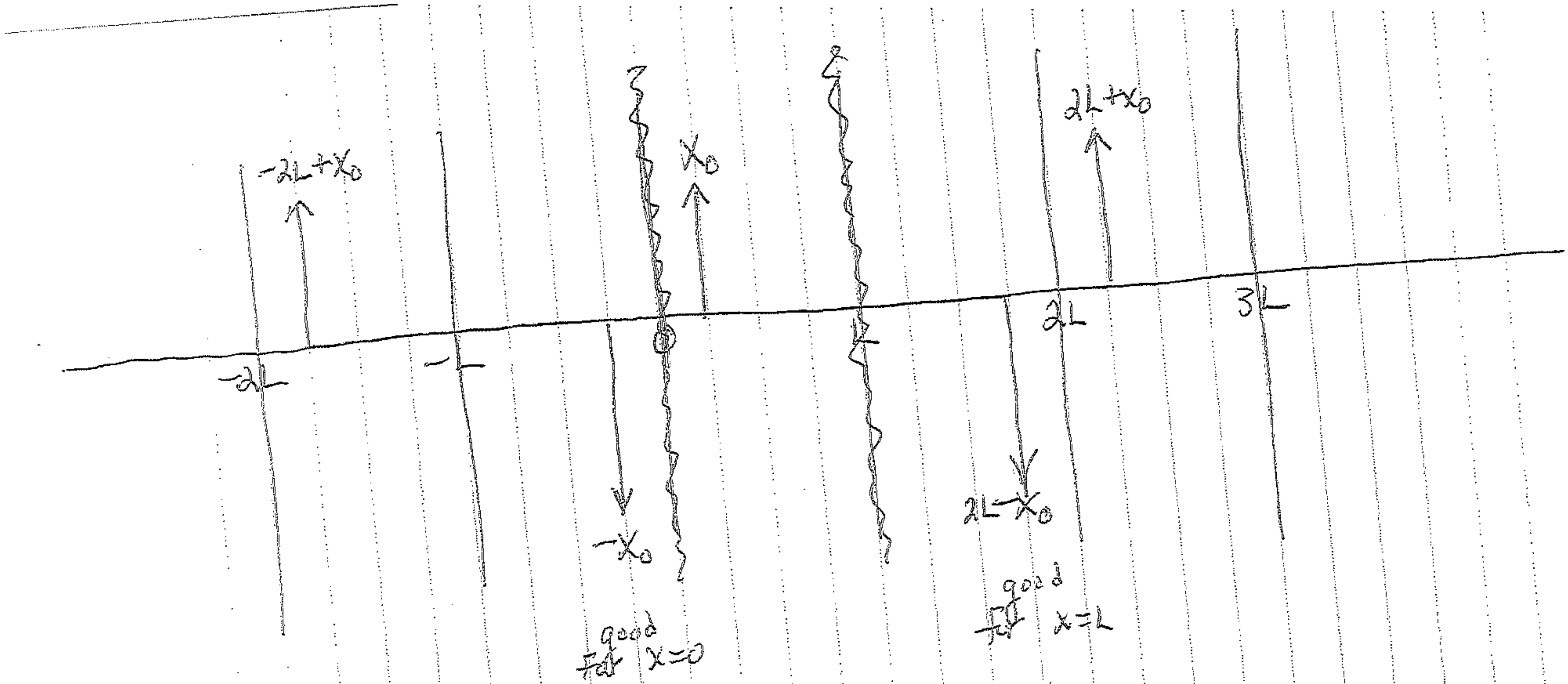
The boundary conditions $G(0,t) = G(L,t) = 0$ will be satisfied, if we take

* positive sources at $x = x_0 + 2Ln$

  n integer $-\infty < n < \infty$

* negative sources at $x = -x_0 + 2nL$

  n integer $-\infty < n < \infty$



Need positive sources at x_0, x_0+2L, x_0+4L, $-2L+x_0$, etc.

Need negative sources at $-x_0$, $-2L-x_0$, $2L-x_0$, $4L-x_0$, etc.

$$G(x_0, t_0 ; x, t) = \frac{1}{\sqrt{4\pi K(t-t_0)}} \sum_{n=-\infty}^{\infty} \Bigg\{$$

$$\exp\left[\frac{-(x-x_0-2nL)^2}{4K(t-t_0)}\right]$$

$$- \exp\left[\frac{-(x+x_0-2nL)^2}{4K(t-t_0)}\right]\Bigg\} \qquad t > t_0$$

$$G(x_0, t_0 ; x, t) = 0 \qquad t < t_0$$