The equation for vorticity $\Omega = \nabla \times \mathbf{u}$:

$$\frac{d\Omega}{dt} + (\mathbf{u} \cdot \nabla) \Omega - (\Omega \cdot \nabla) \mathbf{u} = \nu \nabla^2 \Omega$$

For time-dependent 1D-1C shear flows we have a balance between $\frac{d\Omega}{dt}$ and $\nu \nabla^2 \Omega$ leading to diffusion of vorticity; vorticity spreads over a distance $L = O((\nu t)^{1/2})$.

Let's add a top wall to the problem we did last class:

$$u = 0$$

$$u = u_0 x \quad t > 0$$

At time $t=0$, the bottom wall is set impulsively in motion with speed $u_0$.

$u = u(y, t)$ satisfies $\frac{du}{dt} = \nu \frac{d^2 u}{dy^2}$ (no pressure gradient)
\[ u(y, 0) = 0 \quad 0 < y < h \]
\[ u(0, t) = u_0 \quad t > 0 \quad u(h, t) = 0 \quad t > 0 \]

and we know that there is a steady-state solution \( u_{\text{steady}}(y) = u_0 \left(1 - \frac{y}{h}\right) \)

where this is the linear Couette profile with the appropriate boundary conditions (zero pressure gradient).

To find the transient behavior, let
\[ u(y, t) = u_{\text{steady}}(y) + u_1(y, t) \]

\[ \frac{1}{\nu} \frac{\partial (u_0 + u_1)}{\partial t} = \frac{1}{\nu} \frac{\partial^2 (u_0 + u_1)}{\partial y^2} \implies \frac{\partial u_1}{\partial t} = \nu \frac{\partial^2 u_1}{\partial y^2} \]
\[ u(y,0) = 0 = u_0 \left( 1 - \frac{y}{h} \right) + u_1 \quad 0 < y < h \]

\[ \Rightarrow u_1 = -u_0 \left( 1 - \frac{y}{h} \right) \quad 0 < y < h \]

\[ u(0,t) = U_0 = u_0 + u_1 \Rightarrow u_1(0,t) = 0 \quad t > 0 \]

\[ u(h,t) = 0 = 0 + u_1 \Rightarrow u_1(h,t) = 0 \quad t > 0 \]

Now there is an intrinsic length scale \( l = h \), so the similarity solution approach will not work, but homogeneous boundary conditions for \( u_1(y,t) \) suggests the separation of variables:

Let \( u_1(y,t) = S(y) P(t) \Rightarrow \frac{dP(t)}{dt} = -\sigma^2 y P(t) \quad \frac{d^2 S(y)}{dy^2} = -\sigma^2 S(y) \]

\[ S(0) = S(h) = 0 \]

where the separation constant \(-\sigma^2\) is negative for exponential decay (physically reasonable and the only mathematically consistent choice).

\[ \Rightarrow S_n(y) = A_n \sin \left( \sigma_n y \right) \quad \sigma_n^2 = \frac{n^2 \pi^2}{h^2} \]

\[ P_n(t) = B_n \exp \left( -\frac{n^2 \pi^2}{h^2} \nu t \right) \]
\[ u(y, t) = \sum_{n=1}^{\infty} C_n \exp \left( -\frac{n^2\pi^2}{h^2} t \right) \sin \frac{n\pi y}{h} \]

\[ u(y, 0) = -u_0 \left( 1 - \frac{y}{h} \right) = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi y}{h} \]

and by orthogonality

\[ C_n = -\frac{2}{h} \int_0^h u_0 \left( 1 - \frac{y}{h} \right) \sin \frac{n\pi y}{h} \, dy \]

\[ u(y, t) = u_0 \left( 1 - \frac{y}{h} \right) - \frac{2u_0}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \exp \left( -\frac{n^2\pi^2}{h^2} t \right) \sin \frac{n\pi y}{h} \]

and for \( t \approx \frac{h^2}{v} \) the flow has almost reached steady state, its vorticity has diffused throughout the layer.

\[ \exp \left( -\frac{n^2\pi^2}{h^2} t \right) \approx \exp \left( -\pi^2 \right) \]

For \( n = 1 \), \( \exp (-\pi^2) \approx 5 \times 10^{-4} \)

Then \( \hat{\omega} = -\frac{\partial u(y, t)}{\partial y} \approx -\frac{u_0}{h} \) (the steady state value)
Flow between concentric cylinders

\[ \Omega_1 = \text{rotation rate of inner cylinder} \]
\[ \Omega_2 = \text{rotation rate of outer cylinder} \]

This is a boundary-driven flow (no pressure gradient). Physically we could reason

\[ u = u_0 (r, t) \hat{e}_\theta \]

\[ \{ \text{From the general case} \} \]

\[ u = u_r \hat{e}_r + u_\theta \hat{e}_\theta + u_z \hat{e}_z \]
and remember:

\[ \frac{\partial e_r}{\partial \theta} = e_\theta, \quad \frac{\partial e_\theta}{\partial \theta} = -e_r, \quad \frac{\partial e_z}{\partial \theta} = 0 \]

\[ \nabla \cdot \mathbf{u} = 0 \] (automatically satisfied here)

\[ \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla p + \nu \nabla^2 \mathbf{u} \]

\[ \nabla = \left( \frac{\partial}{\partial r} \hat{e}_r, \frac{1}{r} \frac{\partial}{\partial \theta} \hat{e}_\theta, \frac{1}{r \sin \theta} \frac{\partial}{\partial z} \hat{e}_z \right) \]

\[ (\mathbf{u} \cdot \nabla) \mathbf{u} = u_\theta \frac{1}{r} \frac{\partial}{\partial \theta} \left[ u_\theta(r,t) \hat{e}_\theta \right] \]

\[ = \frac{u_\theta^2 \hat{e}_\theta}{r} = -\frac{u_\theta^2}{r} \hat{e}_r \]

\[ \nu \nabla^2 \mathbf{u} = \nu \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial z^2} \right) \left[ u_\theta(r,t) \hat{e}_\theta \right] \]

with term \[ \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \theta^2} \left( u_\theta \hat{e}_\theta \right) = \frac{u_\theta}{r^2} \frac{\partial}{\partial \theta} \frac{\partial}{\partial \theta} \hat{e}_\theta \]

\[ = -\frac{u_\theta}{r^2} \frac{\partial}{\partial \theta} \hat{e}_r = -\frac{u_\theta}{r^2} \hat{e}_\theta \]
So \( \nabla^2 u = \nu \left( \frac{1}{r^2} \frac{d}{dr} \left( r \frac{du}{dr} \right) - \frac{u}{r^2} \right) \hat{e}_6 \)

\( \Rightarrow \) Navier Stokes becomes

\[ \hat{e}_r : \quad \frac{\nabla^2 u_r}{r} = -\frac{1}{\rho} \frac{dp}{dr} \]

\[ \hat{e}_\theta : \quad \frac{\partial u_\theta}{\partial t} = -\frac{1}{\rho} \frac{1}{r} \frac{dp}{d\theta} + \nu \left( \frac{1}{r^2} \frac{d}{dr} \left( r \frac{du_\theta}{dr} \right) - \frac{u_\theta}{r^2} \right) \]

\[ \hat{e}_z : \quad 0 = -\frac{1}{\rho} \frac{dp}{dz} \]

\( \hat{e}_z : \) \( p \) is independent of \( z \)

\( \hat{e}_\theta : \) since \( u_\theta = u_\theta(r,t) \), then

\[ \frac{dp}{d\theta} = \mathcal{P}(r,t) \]

\( \Rightarrow \) \( p = \mathcal{P}(r,t) \theta + F(r,t) \)

and by periodicity \( p(\theta) = p(2\pi) \)

\[ p(\theta,t) = p(2\pi,t) \]

\( \Rightarrow \) \( \mathcal{P}(r,t) = 0 \)
Then \[
\frac{du_0}{dt} = \nu \left( \frac{\partial^2 u_0}{\partial r^2} + \frac{1}{r} \frac{\partial u_0}{\partial r} - \frac{u_0}{r^2} \right)
\]

From which we find \( u_0(r,t) \), then use the \( \hat{e}_r \)-equation to find \( p(r,t) = \hat{F}(r,t) \).

In the steady-state, \( \frac{du_0}{dt} = 0 \) and the general solution is \( u_0(r) = Ar + B \).

This is an equidimensional operator

\[
r^2 \frac{d^2 u_0}{dr^2} + r \frac{du_0}{dr} - u_0 = 0
\]

so plug in \( u_0 = r^a \) \Rightarrow

\[
a(a-1) + a - 1 = 0 \Rightarrow a^2 = 1 \Rightarrow a = \pm 1
\]

For boundary conditions, we have no-slip at each cylinder wall.
\[ u_0(r) = Ar + \frac{B}{r^2} \]

\[ u_0(r=r_1) = \Omega_1 \quad u_0(r=r_2) = \Omega_2 \]

algebra \[ A = \frac{\Omega_2 r_2^2 - \Omega_1 r_1^2}{r_2^2 - r_1^2} \]

\[ B = \frac{(\Omega_1 - \Omega_2) r_2^2 r_1^2}{r_2^2 - r_1^2} \]

Acheson Ch. 9 we will see how this flow is unstable for large \( \Omega_1 \).

\[ \text{check mass conservation :} \quad \nabla \cdot \mathbf{u} = 0 \]

\[ \frac{1}{r} \frac{\partial}{\partial r} (r u_r) + \frac{1}{r} \frac{\partial}{\partial \theta} u_\theta + \frac{\partial u_z}{\partial z} = 0 \quad \checkmark \]
"Spin down" in the infinitely long cylinder

This is a simple time-dependent flow for which we can apply all the previous work.

\[ u = u_0(r, t) \hat{e}_z \]

\[ u_0 = -\lambda r, \quad r \leq a, \quad t = 0 \]
\[ \nabla \times u = 2 \lambda \hat{e}_z \]

(we just showed this form of solution is allowed in steady state)

Now the cylinder is suddenly brought to rest such that

\[ u_0(r=a, t) = 0, \quad t > 0 \]

We can use the previously derived equation

\[ \frac{\partial u_0}{\partial t} = \nu \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u_0}{\partial r} \right) - \frac{\partial^2 u_0}{\partial r^2} \right] \]

with Separation of Variables
\[ \nabla \times \mathbf{u} = \frac{1}{r} \begin{vmatrix} \hat{e}_r & \hat{e}_\theta & \hat{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ u_r & r u_\theta & u_z \end{vmatrix} \]

\[ = \frac{1}{r} \hat{e}_z \left( \frac{\partial}{\partial r} (r u_\theta) - \frac{\partial u_r}{\partial \theta} \right) \]

\[ = \frac{1}{r} \hat{e}_z \frac{\partial}{\partial r} (r^2) = \hat{e}_z 2r \]

Here we start with vorticity and watch it decay.
\[ u_0(r,t) = R(r) S(t) \Rightarrow \]

\[ S' = -\sigma^2 S \quad \Rightarrow \quad R'' + \sigma R' - R = -\sigma^2 \sigma^2 R \]

\[ \text{exponential decay} \quad \text{Bessel eqn. of order 1} \]

\[ \text{Boundary/initial conditions:} \]

\[ u_0(r,0) = R(r) S(0) = \frac{S}{L} \quad \text{(initial condition)} \]

\[ u_0(\infty, t) = 0 = R(\infty) S(t) \quad \text{boundary conditions} \]

\[ u_0(0, t) \text{ bounded} \Rightarrow R(0)S(t) \text{ bounded} \]

\[ R(r) = J_1(\sigma r) \quad \text{bounded} \]

\[ J_1(\sigma a) = 0 \quad \text{gives the values} \]

\[ \sigma_n = \frac{\lambda_n}{a} \quad \text{where } \lambda_n \text{ are the} \]

\[ \text{zeros of } J_1(x), \lambda_n \text{ known} \]

\[ \lambda_1 < \lambda_2 < \ldots < \lambda_n < \ldots \]
$$U_0(r,t) = R(r) S(t)$$

$$= \sum_{n=1}^{\infty} A_n J_1 \left( \frac{\lambda_n}{\alpha} r \right) \exp \left( -\gamma \frac{\lambda_n^2}{\alpha^2} t \right)$$

and by orthogonality

$$A_n = \frac{1}{\int_0^a r J_1 \left( \frac{\lambda_n}{\alpha} r \right) J_1 \left( \frac{\lambda_n}{\alpha} r \right) \, dr} \int_0^a \left( \frac{\lambda_n}{\alpha} r \right) J_1 \left( \frac{\lambda_n}{\alpha} r \right) \, dr$$

Again we see the combination $\frac{\gamma^2 t}{\alpha^2} = \frac{\gamma^2 t}{L^2}$

leading to the vorticity diffusion time

$$t = O \left( \frac{L^2}{\gamma} \right)$$

when $t = \frac{a^2}{\eta \lambda_1^2}$, $1.2 \approx 3.83$, then the function is $\frac{1}{2}$ times the initial value.

But we cannot use this to estimate the spin down time in a cup of coffee because there the bottom boundary layer is important!