Chapter 4: Classical Aerofoil Theory
Chapter 5: Vortex Motion

Acheson mentions already in Chapter 1, the Kutta-Joukowski Theorem (1.35) pertaining to the lift force on a 2D aerofoil.

The set-up in the following

![Diagram of an aerofoil with flow](image)

Uniform Flow at $x \rightarrow -\infty$

2D aerofoil: constant density $\rho$, angle of attack $\alpha$

The theorem says that $L = \rho u_0 |M|$

where $L$ is lift and $M$ is circulation

$L$ is the force perpendicular to the Freestream direction $\hat{z}$.
What exactly is $\alpha$?

Define the chord length as the longest distance for a straight line within the aerofoil.

Then $\alpha$ is the angle from the free-stream direction $\hat{x}$ to this line.

$\Gamma = \Gamma(\alpha)$

$\Gamma$ also depends on the shape of the aerofoil.
\[ \Gamma \text{ is circulation about a closed curve } C \text{ lying in the fluid region } \] 
\[ \Gamma = \oint_C \mathbf{u} \cdot d\mathbf{x} \]

So, for example, consider the following three scenarios:

\[ \Gamma_1 \]

\[ \Gamma_2 \]

\[ \Gamma_3 \]

\( \circ \) indicates a stagnation point \( \mathbf{u} = 0 \)
The inviscid, irrotational, incompressible ADAC theory tells us

\[ \mathbf{F} = -p \mathbf{n} \]

but does not give us \( F \). To determine the circulation, we need to invoke viscosity and

**The Kutta condition** says that there should be no sharp gradients in the velocity (viscosity does not allow sharp gradients), therefore the value of \( F \) is such that the stagnation point is at the sharp trailing edge of the aerofoil.

The inviscid, irrotational, incompressible, ADAC theory also gives drag

\[ D = 0 \]

what drag is the force on the aerofoil in the direction of the freestream.
Let's start with a more modest goal: try to understand flow around a circular cylinder with and without circulation.

\[ u = u_0 \hat{x} \]

Case \( \gamma = 0 \)

Then we will map the circle to the aerofoil shape.

Why start with rotational flow?

- We already discussed the fact that we can solve a linear pde for the stream function \( \Psi \).

- Consider the vorticity equation

\[
\frac{d\omega}{dt} + (u \cdot \nabla) \omega = (\omega \cdot \nabla) u + \nu \Delta \omega
\]

- Completely general: find by taking the curl of the equation; fixed \( u \) and using chain rule.

- And we consider the incompressible problem \( \nu = 0 \).
2D-2C Flow in Cartesian coordinate

\[
\begin{vmatrix}
\frac{\partial \hat{x}}{\partial x} & \frac{\partial \hat{y}}{\partial x} & \frac{\partial \hat{z}}{\partial x} \\
\frac{\partial \hat{x}}{\partial y} & \frac{\partial \hat{y}}{\partial y} & \frac{\partial \hat{z}}{\partial y} \\
\frac{\partial \hat{x}}{\partial z} & \frac{\partial \hat{y}}{\partial z} & \frac{\partial \hat{z}}{\partial z}
\end{vmatrix}
\begin{bmatrix}
\hat{x} \\
\hat{y} \\
\hat{z}
\end{bmatrix}
\begin{bmatrix}
\hat{u} \\
\hat{v} \\
\hat{w}
\end{bmatrix}
= \hat{z} \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)
\]

\[
u = u(x,y)\hat{x} + v(x,y)\hat{y} + \hat{z} \hat{w}
\]

\[
w = w\hat{z} \implies (\hat{u} \cdot \nabla) w = 0
\]

and the same result for polar coordinates.

\[
\frac{\partial w}{\partial t} + (\hat{u} \cdot \nabla) w = 0
\]

Now for steady flow such as an aerofoil problem:

\[(\hat{u} \cdot \nabla) w = 0\] and there is no change of \(w\) in the direction of a streamline (tangent to \(\hat{u}\)) \(\implies\) no change in \(w\) along a streamline.
Now since \( \omega = 0 \) at \( x \to -\infty \), then \( \omega = 0 \) everywhere.

For the inviscid problem

Our set up is now

\[ \hat{e}_z = \hat{z} \text{ out of page} \]
Recall that for incompressible flow (2D-2C)
\[ \nabla \cdot u = 0 \text{ satisfied if } u = \nabla \times \psi \hat{k} \text{ where } \psi \text{ is the streamfunction} \]

\[ [u \cdot \nabla] \psi = 0 \Rightarrow \psi \text{ constant along streamlines} \]

Then if the flow is also irrotational with

\[ \nabla \times u = 0 \Rightarrow \nabla \times (\nabla \times \psi \hat{k}) = 0 \Rightarrow \nabla^2 \psi = 0 \]

For irrotational flow, \( \nabla \times u = 0 \) satisfied

\[ \text{if } u = \nabla \psi, \text{ where } \psi \text{ is a potential.} \]

Then if the flow is also incompressible with

\[ \nabla \cdot u = 0 \Rightarrow \nabla \cdot (\nabla \psi) = 0 \Rightarrow \nabla^2 \psi = 0 \]

Notice here the + sign convention for the potential flow following Acheson Chapter 4!
So we can solve Laplace's Eqn. for $\phi$ or $\psi$ with appropriate boundary conditions!

In Cartesian coordinates $u = (u, v) = (u\hat{x}, v\hat{y})$

$$u = \frac{\partial \psi}{\partial y} = \frac{\partial \psi}{\partial x}, \quad v = -\frac{\partial \psi}{\partial x} = \frac{\partial \phi}{\partial y}$$

In Polar coordinates $u = (u_r \hat{r}, u_\theta \hat{\theta})$

$$u_r = \frac{1}{r} \frac{\partial \psi}{\partial \theta} = \frac{\partial \psi}{\partial r}, \quad u_\theta = -\frac{\partial \psi}{\partial r} = \frac{1}{r} \frac{\partial \phi}{\partial \theta}$$

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Some "building block" or "toy" flows:

1) Uniform Flow $\Rightarrow u = u_0 \hat{x}$

in an unbounded domain

$\phi(x, y) = u_0 y \quad \psi(x, y) = u_0 x$

unbounded so no boundary conditions necessary
(2) Source Flow

\[ u_r = \frac{q}{2\pi} \frac{1}{r}, \quad u_\theta = 0 \]

velocity is radial with a singularity at \( r = 0 \)

\[ \psi = \frac{q \theta}{2\pi}, \quad \varphi = \frac{q}{2\pi} \ln r \]

\( q \) is the area flow rate \( [q] = \left[ \frac{\text{in}^2}{\text{sec}} \right] \)

with \( q = \int_0^{2\pi} u_r r d\theta \)

(3) The irrotational vortex

\[ u_r = 0, \quad u_\theta = \frac{q}{2\pi r} \]

\[ \psi = -\frac{q}{2\pi} \ln r, \quad \varphi = \frac{q \theta}{2\pi} \]
The irrotational vortex

\[ M = \oint_{\Gamma} \mathbf{u} \cdot d\mathbf{r} = \oint_{\Gamma} u_\theta \, r \, dr \]

is circulation; \( r = 0 \) is a singularity again.

It is easy to check that all these flows have \( \psi, \Psi \) that satisfy

\[ \nabla^2 \psi = \nabla^2 \Psi = 0 \]

Since the equations for \( \psi, \Psi \) are linear, we can add solutions and the result must also satisfy the differential equation.

e.g., let's add a sink + irrotational vortex

\[ \Psi = -\frac{1}{2\pi} \ln r - \frac{g}{2\pi} \theta \]

vortex  sink
\[ \psi = \frac{\rho}{\eta} \theta - \frac{\rho}{\beta_{11}} \ln r \]

We can verify that flow over a cylinder with circulation \( \Gamma \) has streamfunction

\[ \psi = u_0 r \left( 1 - \frac{a^2}{r^2} \right) \sin \theta - \frac{r}{\alpha_1} \ln r \]

\[ u_r = \frac{1}{r} \frac{\partial \psi}{\partial \theta} = \frac{1}{r} u_0 \left( r - \frac{a^2}{r} \right) \cos \theta \]

\[ u_\theta = -\frac{1}{r} \frac{\partial \psi}{\partial r} = -u_0 \sin \theta \left( 1 + \frac{a^2}{r^2} \right) + \frac{r}{\alpha_{11}} \frac{1}{r} \]

\( u_0, a, \Gamma \) are parameters; \( u_0 \) is the freestream velocity; \( a \) is the radius of the cylinder; \( \Gamma \) is circulation.
\[
\begin{align*}
M = 0 & \quad u_r = U_0 \left(1 - \frac{a^2}{r^2}\right) \cos \theta \\
& \quad u_\theta = -U_0 \left(1 + \frac{a^2}{r^2}\right) \sin \theta
\end{align*}
\]

\[
\begin{align*}
\text{Stagnation points} & \quad r = a, \theta = \pm \theta_0 : \quad u_r = 0, \quad u_\theta = 0 \\
& \quad r = a, \theta = 0 : \quad u_r = 0, \quad u_\theta = 0 \\
& \quad r = a : \quad u_r = 0, \quad u_\theta = -2U_0 \sin \theta \\
& \quad (\text{check } \theta = \frac{\pi}{2}, \frac{3\pi}{2}) \\
& \quad r \to \infty, \theta = \frac{\pi}{2} : \quad u_r = 0, \quad u_\theta = -U_0 \\
& \quad r \to \infty, \theta = \pi : \quad u_r = -U_0, \quad u_\theta = 0 \\
& \quad r \to \infty, \theta = 0 : \quad u_r = \frac{U_0}{r}, \quad u_\theta = 0 \\
\end{align*}
\]

e tc. \quad \text{Symmetric Flow with no circulation}
\[ \theta > 0 \Rightarrow \text{some counterclockwise circulation} \]

\[ M < 0 \Rightarrow \text{some clockwise circulation} \]
Calculate lift and drag using Bernoulli for steady, inviscid flow.

\[
\text{drag} = \text{force in horizontal direction } (x) \quad \text{lift} = \text{force in vertical direction } (y)
\]

\[
\frac{p_1}{\rho} + \frac{1}{2} u_1 \cdot u_1 + gy_1 = \frac{p_2}{\rho} + \frac{1}{2} u_2 \cdot u_2 + gy_2 = \text{c}
\]

Neglect gravity and take \( r = \infty \) and \( r = a \):

\[
\mathbf{u} = u_r \mathbf{\hat{e}}_r + u_\theta \mathbf{\hat{e}}_\theta
\]

\[
= u_0 \left( 1 - \frac{a^2}{r^2} \right) \cos \theta \mathbf{\hat{e}}_r + \left( -u_0 \sin \theta (1 + \frac{a^2}{r^2}) + \frac{r}{a \pi r} \right) \mathbf{\hat{e}}_\theta
\]

\[
p_\infty + \frac{\rho}{2} \left\{ -u_0^2 \cos^2 \theta + u_0^2 \sin^2 \theta \right\} =
\]

\[
p_a + \frac{\rho}{2} \left\{ -2u_0 \sin \theta + \frac{1}{2\pi a} \right\} \theta^2
\]
\[ p_a = p_0 + \frac{1}{2} \rho u_0^2 - \frac{1}{2} \rho \left( -2 u_0 \sin \theta + \frac{\dot{V}}{2 \pi a_0} \right)^2 \]

\[ \text{Drag } D = \int_A p_a (-\hat{n}) \, dA = i \]

where \( A \) is the surface of the cylinder at fixed \( r = a \)

\[ \hat{n} \cdot \hat{z} = -\cos \theta \]

\[ dA = a \, d\theta \, dz \quad \hat{z} \text{ out of the page} \]

but \( b \) = width of the cylinder out of the page

\[ D = b \int_0^{2\pi} p_a (-\cos \theta) \, a \, d\theta = 0 \]
D'Alembert's paradox: \( D = 0 \) is the first reason why we need to include viscosity!

\[
L = b \int_0^{2\pi} p_a (-\hat{n}) \, dA \cdot \hat{j} \\
\text{at } r = a \text{ fixed}
\]

\[
L = b \int_0^{2\pi} p_a (-\sin \theta) a \, d\theta = -\rho u_0^2 b
\]

\[
\Rightarrow \frac{L}{b} = -\rho u_0^2 \text{ (lift per unit width)}
\]

Do the calculations yourself!

Now we need to deduce the stream function...