What happens if we keep the constant of integration in the integrating factor? 

Define \( D u(x) = \exp \left[ \int p(x) \, dx + c \right] \)

Then \( \frac{d}{dx} \left[ D u y \right] = D u q \)

\( D u y = \int D u q \, dx + F \)

\( y = \frac{1}{D u} \int D u q \, dx + \frac{F^*}{D u} \)

\( y(x) = \frac{1}{m(x)} \int m(x) q(x) \, dx + \frac{F^*}{m(x)} \)

\( F^* = \frac{F}{D} \) constant

Same as before.

This explicit formula depends on \( p(x), q(x) \) and can have singularities, etc. where \( p(x), q(x) \) have singularities \( \Rightarrow \)
For linear, 1st-order equations, there is a simpler thm. of existence & uniqueness

Thm of E & U for linear, 1st-order eqns.

Given

\[ y'(x) + p(x) y(x) = q(x) \quad y(x_0) = y_0 \]

If \( p(x), q(x) \) are real continuous in
\[ \alpha < x < \beta \] with \( \alpha < x_0 < \beta \)

Then there is a unique solution in (at least)
\[ \alpha < x < \beta \]

Notes: The region of validity of the solution can be bigger than \( (\alpha, \beta) \)

In the form
\[ \frac{dy}{dx} = f(x, y) \]

now \( f(x, y) = -p(x) y(x) + q(x) \)

\[ \frac{dF(x, y)}{dy} = -p(x) \]
* we look for an open interval where $p(x), q(x)$ are real continuous

* problem points are singularities in the coefficient $p(x)$ and the forcing function $q(x)$, etc.

* helps to understand why we check $\frac{df(x,y)}{df}$ in the general thm.
Example 1 \[ xy' + 2y = 3x \quad y(1) = 1 \]

Standard Form \[ y' + \frac{2}{x} y = 3 \quad y(1) = 1 \]

\( p(x) \) is singular at \( x = 0 \) so we might conclude \( x > 0 \),

but the solution is \( y = x \), \( -\infty < x < \infty \).

Example 2 \[ xy' + 2y = 3x \quad y(1) = 5 \]

\[ y = x + \frac{4}{x^2} \quad x > 0 \]
Autonomous systems; Population Dynamics; Critical Points; Stability

Autonomous ODEs have the form \( \frac{dy}{dx} = f(y) \).

In this case, graphical techniques are powerful illustration using a hierarchy of population growth models.

We will make sketches for 2 types of plots: (i) the direction field, and (ii) \( \frac{dN(t)}{dt} \) vs. \( N(t) \).

Where \( N(t) \) is the population of a species at time \( t \).

Model 1: The rate of change of \( N(t) \) is proportional to \( N(t) \) itself.

\[ \frac{dN(t)}{dt} \propto N(t) \]

Small population \( \rightarrow \) small growth

Large population \( \rightarrow \) large growth
\[
\frac{dN(t)}{dt} = \alpha N(t) \quad \alpha > 0 \text{ constant growth rate}
\]

\[N(0) = N_0\]

* The slope \(\frac{dN}{dt}\) is positive for all \(N > 0 \Rightarrow\) the population grows for all time for \(N_0 > 0\)

* Critical points: are points at which \(\frac{dN(t)}{dt} = 0\) when you are at a critical point you stay at the critical point because the rate of change is zero. Also called equilibrium points.
* The only critical point is \( N=0 \). If the population is initially zero, it stays zero for all time. However, if the population is infinitesimally larger than zero, then it grows for all time.

\[
N(0) = E^+ \Rightarrow \text{growth for all time}
\]

Thus the critical point \( N=0 \) is \text{unstable}.

A rough sketch of the direction field:

\[
N = \text{exponential growth } N(t) = N_0 \exp(\alpha t)
\]
Model 2: Logistic Growth

Instead of a constant growth rate \( \alpha \), let the growth rate depend on \( N(t) \):

\[
\frac{dN(t)}{dt} = F(N)N
\]

Sensible to take \( F(N) \to r > 0 \) for \( N \to 0 \)

\( F(N) \) negative for \( N \to \infty \)

* exponential growth for small \( N(t) \)
* wars, famines, etc. will cause the species to reduce its numbers if \( N(t) \) becomes too large

Let \( F(N) = (r - aN) \) for \( r > 0, a > 0 \)

\[
= r(1 - \frac{a}{r} N) = r(1 - \frac{N}{K})
\]

with \( K = \frac{r}{a} > 0 \) constant.

The logistic growth model

\[
\frac{dN(t)}{dt} = r \left( 1 - \frac{N}{K} \right) N
\]
We could solve this analytically using separation, but let's solve it graphically.

Consider \( \frac{dN(t)}{dt} \) vs. \( N(t) \)

* \( \frac{dN(t)}{dt} > 0 \) for \( N < K \), \( 0 < N < K \) (growth)
* \( \frac{dN(t)}{dt} < 0 \) for \( N > K \) (decay)

* critical points are \( N=0 \), \( N=K \)

![Direction Field](image)

Curvature ignored here.
Stability

$N=0$ is unstable. If you move $\varepsilon$ away from $N=0$, there is growth away from $N=0$.

$N=k$ is stable. If you move $\varepsilon$ away from $N=k$, there is decay or growth back to the level $N=k$.

$N(t) = k$ is the saturation level.

Model 3: Model with a critical threshold
\[
\frac{dN(t)}{dt} = -r \left(1 - \frac{N}{T}\right) N
\]

- $\frac{dN}{dt} < 0$ for $0 < N < T$
- $\frac{dN}{dt} > 0$ for $T < N$

- Critical points are $N=0, N=T$
\[
\frac{dN}{dt} \text{ vs } N
\]

**Direction Field**

Stability: \( N = 0 \) is stable, \( N = T \) is unstable

*If the initial population \( 0 < N_0 < T \), then the species eventually dies out.*

*The species only exists indefinitely if \( N_0 \) is above the threshold \( T \).*
Model 4) Logistic Growth with a Threshold

\[
\frac{dN(t)}{dt} = -r \left(1 - \frac{N}{T}\right) \left(1 - \frac{N}{K}\right) N
\]

\[K > T\]

- Critical points: \(N = 0\), \(N = K\), \(N = T\)

- \(0 < N < T\) \(-++\) \(\Rightarrow \frac{dN}{dt} \leq 0\)

- \(T < N < K\) \(-+-+\) \(\Rightarrow \frac{dN}{dt} > 0\)

- \(K < N\) \(---\) \(\Rightarrow \frac{dN}{dt} < 0\)

\[\frac{dN}{dt} \text{ vs } N\]
Direction Field

Stability:

- $N = 0$ stable
- $N = T$ unstable
- $N = K$ stable

$N = T$ is a threshold

$N = K$ is a saturation level

The most realistic model yet!