Math 322 Lecture 6 Continuing ...

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} \quad 0 < x < L, \quad t > 0$$

$$u(0,t) = 0, \quad u(L,t) = 0, \quad u(x,0) = f(x)$$

$$u(x,t) = \phi(x)\, G(t) \implies$$

x - dependence $\quad \dfrac{d^2\phi(x)}{dx^2} = -\lambda \phi(x)$

$$\phi(0) = \phi(L) = 0$$

and we find $\quad \lambda_n = \dfrac{n^2\pi^2}{L^2} \quad n = 1, 2, 3$

$$\phi_n(x) = \sin\left(\frac{n\pi x}{L}\right)$$

Now t-dependence $\quad \dfrac{dG(t)}{dt} = -\lambda k G(t)$;

For each λ_n: $\quad \dfrac{dG_n(t)}{dt} = -\lambda_n k G_n(t)$

$\uparrow$

linear, 1st-order, homogeneous

with soln. $\quad G_n(t) = C_n \exp(-\lambda_n k t)$

Thus for each λ_n:

$$u_\lambda(x,t) = \phi_\lambda(x)\, G_\lambda(t) \quad \text{or} \quad u_n(x,t) = \phi_n(x)\, G_n(t)$$

$$u_n(x,t) = B_n \sin\left(\sqrt{\lambda_n}\, x\right) \exp\left(-\lambda_n k t\right)$$

$$= B_n \sin\left(\frac{n\pi x}{L}\right) \exp\left(-\frac{n^2\pi^2}{L^2} k t\right)$$

with $B_n = b_n c_n$

By Superposition, the General Soln is $u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) \exp\left(\frac{-n^2\pi^2}{L^2} k t\right)$$

Notice that, thus far, we have only determined the solution up to the arbitrary coefficients B_n

Now use initial condition $u(x,0) = f(x)$

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) = f(x)$$

$\boxed{\text{Example 1}}$ $f(x) = 32 \sin\left(\frac{5\pi x}{L}\right)$

$\Rightarrow$ we only have $n=5$ with $B_5 = 32$

$$u(x,t) = 32 \sin\left(\frac{5\pi x}{L}\right) \exp\left(-\frac{25\pi^2}{L^2} k t\right)$$

satisfies the PDE, 2 b.c.s, initial condition

$\boxed{\text{Check}}$

$\boxed{\text{Example 2}}$ $f(x) = 32 \sin \frac{5\pi x}{L} + 10 \sin \frac{7\pi x}{L}$

Now we see for $n=5$, $B_5 = 32$
for $n=7$, $B_7 = 10$

no other values of n $\Rightarrow$

$$u(x,t) = 32 \sin\left(\frac{5\pi x}{L}\right) \exp\left(\frac{-25\pi^2}{L^2} k t\right) + 10 \sin\left(\frac{7\pi x}{L}\right) \exp\left(\frac{-49\pi^2}{L^2} k t\right)$$

Satisfies the PDE, b.c.s, initial condition

$\boxed{\text{Check}}$

For more general functions, we might need <u>many</u> or <u>all</u> $n = 1, 2, 3, \ldots$

* If we have an initial condition that contains many or all the sine functions with $n = 1, 2, 3 \ldots$

* If $f(x)$ projects onto many or all of the sine functions

e.g. $f(x) = x(x-L)$

{ notice I choose $f(x)$ to satisfy the b.c.s $f(0) = f(L) = 0$; do we need this ? }

$\exists$ a class of functions $f(x)$ that can be approximated by

$$f(x) \approx \sum_{n=1}^{M} B_n \sin\frac{n\pi x}{L} \qquad 0 \le x \le L$$

with improved approximation as $M \to \infty$ such that the series will converge as $M \to \infty$

We need to specify the class of functions! $\boxed{\text{Ch. 3}}$

For now, lets assume that our initial condition $f(x)$ can be written as a superposition of sine functions

$$f(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L}$$ so that the soln. to our heat conduction problem is

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) \exp\left(-k \frac{n^2\pi^2}{L^2} t\right)$$

satisfies the PDE, b.c.s, initial condition

How do we find the B_n's, e.g. if $f(x) = x(x-L)$?

Need $\boxed{\text{Orthogonality of Sines}}$ 2.3.6

It is "easy" to check that

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx = \begin{cases} 0 & m \neq n \\ \frac{L}{2} & m = n \end{cases}$$

e.g. check using trig identities (hw)

Now use these formulas to find B_n for

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} \qquad 0 \le x \le L$$

* Multiply both sides by $\sin\frac{m\pi x}{L}$

$$\sin\frac{m\pi x}{L} f(x) = \sin\frac{m\pi x}{L} \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}$$

$$= \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} \sin\frac{m\pi x}{L}$$

* Now integrate from 0 to L

$$\int_0^L \sin\frac{m\pi x}{L} f(x)\,dx = \int_0^L \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} \sin\frac{m\pi x}{L}\,dx$$

* If the series were finite, we could interchange the order of integration and summation. Lets <u>assume</u> we can do that for the infinite series.

$$\int_0^L \sin\frac{m\pi x}{L} f(x)\,dx = \sum_{n=1}^{\infty} \int_0^L B_n \sin\frac{n\pi x}{L} \sin\frac{m\pi x}{L}\,dx$$

* The integral is nonzero only for $m=n$ $\Rightarrow$

$$\int_0^L \sin\frac{m\pi x}{L} F(x)\,dx = B_m \int_0^L \sin^2\left(\frac{m\pi x}{L}\right) dx = B_m \frac{L}{2}$$

* Finally $\quad B_m = \frac{2}{L}\int_0^L F(x) \sin\frac{m\pi x}{L}\,dx$

$$\left\{ \text{same as} \quad B_n = \frac{2}{L}\int_0^L F(x) \sin\frac{n\pi x}{L}\,dx \right\}$$

* B_m can be evaluated numerically if not analytically

Orthogonality

Recall that 2 vectors $\underline{A}$ and $\underline{B}$ are said to be orthogonal if $\underline{A} \circ \underline{B} = 0$
(perpendicular)

In Cartesian coordinates $\quad \underline{A} = a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}$

$$B = b_1 \hat{i} + b_2 \hat{j} + b_3 \hat{k}$$

$$\underline{A} \cdot \underline{B} = a_1 b_1 + a_2 b_2 + a_3 b_3 = \sum_{K=1}^{3} a_K b_K$$

For functions $A(x)$, $B(x)$ $0 \le x \le L$ the analogy to the dot product is the inner product

$$\int_0^L A(x)\,B(x)\,dx = 0 \quad \text{for orthogonality}$$

An orthogonal set of functions: each member of the set is orthogonal to every other member of the set in some domain

e.g. $\sin\left(\frac{n\pi x}{L}\right)$, $\sin\left(\frac{m\pi x}{L}\right)$ $\quad 0 \le x \le L$

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = \begin{cases} 0 & m \ne n \\ \frac{L}{2} & m = n \end{cases}$$