When it's not easy to solve an ODE analytically, we can try to find an approximate solution numerically.

Consider \( \frac{dy(t)}{dt} = F(t, y) \) \( y(t_0) = y_0 \)

Let \( \tilde{y}_n = \tilde{y}(t_n) \) denote the approximate solution at the discrete times \( t_n \). For simplicity let's take a constant step size

\( t_{n+1} - t_n = h, \ h > 0 \)

**The Forward Euler Method:**

\[
\frac{\tilde{y}_{n+1} - \tilde{y}_n}{t_{n+1} - t_n} = \frac{\tilde{y}_{n+1} - \tilde{y}_n}{h} = F(t_n, \tilde{y}_n) \quad n \geq 0
\]

In the limit \( h \to 0 \), we get back the ODE

\[
\lim_{h \to 0} \left[ \frac{\tilde{y}_{n+1} - \tilde{y}_n}{h} = F(t_n, \tilde{y}_n) \right] \Rightarrow \frac{dy(t)}{dt} = F(t, y)
\]
\[ \hat{y}_{n+1} = \hat{y}_n + h \hat{f}(t_n, \hat{y}_n) \]
\[ \hat{y}_n = y_0 + h \hat{f}(t_0, y_0) \]
\[ \hat{y}_2 = \hat{y}_1 + h \hat{f}(t_1, \hat{y}_1) \text{ etc.} \]

Words to describe FE:

1. Explicit: \( \hat{y}_{n+1} \) depends only on earlier times and earlier approximations \( \hat{y}_n, \hat{y}_{n-1}, \ldots, y_0 \)
2. One-step: \( \hat{y}_{n+1} \) depends only on \( \hat{y}_n, t_n \)
Example \[ \frac{dy}{dx} = x - y \quad y(0) = 1 \]

Linear, standard form \[ \frac{dy}{dx} + y = x \]

\( \mu(x) = \exp \left[ \int 1 \, dx \right] = e^x \)

\( \frac{d}{dx} \left[ e^x y \right] = e^x x \)

\( e^x y = xe^x - e^x + C \)

\( y(0) = 1 \quad \Rightarrow \quad 1 = -1 + C \quad \Rightarrow \quad C = 2 \)

\[ y = x - 1 + 2e^{-x}, \quad -\infty < x < \infty \]

So we can find the exact solution at any \( x \), e.g.

\( x = 0.4 \quad \Rightarrow \quad y(0.4) \approx 0.74064 \)

Now, let's solve using FE

\[ \hat{y}_{n+1} = \hat{y}_n + hF(x_n, \hat{y}_n) \quad y(0) = 1, \quad x_0 = 0, \quad y_0 = 1 \]
Let's find the solution at $x = 0.4$ using a step size of (a) $h = 0.2$, (b) $h = 0.1$

(a) \[ \# \text{steps} = \frac{x_{\text{final}} - x_0}{h} = \frac{0.4 - 0}{0.2} = 2 \]

1. \text{Step #1} \quad x_0 = 0, \quad x_1 = 0.2

\[ \hat{y}_1 = y_0 + h f(x_0, y_0) = y_0 + h(x_0 - y_0) \]
\[ = 1 + 0.2(0 - 1) = 0.8 \]

\[ \hat{y}_1 = \hat{y}(0.2) = 0.8 \]

1. \text{Step #2} \quad x_1 = 0.2, \quad x_2 = 0.4

\[ \hat{y}_2 = \hat{y}_1 + h f(x_1, \hat{y}_1) = \hat{y}_1 + h(x_1 - \hat{y}_1) \]
\[ = 0.8 + 0.2(0.4 - 0.8) = 0.8 + 0.2(-0.4) \]
\[ = 0.68 \]

\[ \hat{y}_2 = \hat{y}(0.4) = 0.68 \]
(b) $h = 0.1 \quad \#\text{steps} = \frac{x_{\text{final}} - x_0}{h} = \frac{0.4 - 0}{0.1} = 4$

**Step #1**

$x_1 = 0.1$

\[ \tilde{y}_1 = y_0 + h(x_0 - y_0) = 1 + 0.1(0 - 1) = 0.9 \]

**Step #2**

$x_2 = 0.2$

\[ \tilde{y}_2 = \tilde{y}_1 + h(x_1 - \tilde{y}_1) = 0.9 + 0.1(0.1 - 0.9) \]

\[ = 0.9 + 0.1(-0.8) = 0.9 - 0.08 = 0.82 \]

**Step #3**

$x_3 = 0.3$

\[ \tilde{y}_3 = \tilde{y}_2 + h(x_2 - \tilde{y}_2) = 0.82 + 0.1(0.2 - 0.82) \]

\[ = 0.758 \]

**Step #4**

$x_4 = 0.4$

\[ \tilde{y}_4 = \tilde{y}_3 + h(x_3 - \tilde{y}_3) = 0.758 + 0.1(0.3 - 0.758) \]

\[ \tilde{y}_4 = \tilde{y}(0.4) = 0.7122 \]

Exact \[ y(0.4) = 0.4 - 1 + 2e^{-0.4} \approx 0.74064 \]
Example \[ \frac{dy}{dx} = \frac{x^3}{(y+3)^2}, \quad y(-2) = 4 \]

Let's say we want the approximate solution at \( x = 2 \) and we take a step of \( h = 0.1 \)

\[ \text{# steps} = \frac{x_{\text{final}} - x_0}{h} = \frac{2 - (-2)}{0.1} = 40 \]

\[ \hat{y}_{n+1} = \hat{y}_n + h \cdot F(x_n, \hat{y}_n) = \hat{y}_n + h \cdot \frac{x_n^3}{(\hat{y}_n + 3)^2} \]

\[ x_n = x_0 + nh \quad n \geq 1 \]

**Step #1** \( x_1 = -1.9 \)

\[ \hat{y}_1 = \hat{y}(-1.9) = y_0 + h \cdot \frac{x_0^3}{(y_0 + 3)^2} \]

\[ = 4 + 0.1 \cdot \frac{(-2)^3}{(4 + 3)^2} \]

\[ \approx 3.9836735 \]
Let's keep 4 digits (computer keeps 16)

\[ \tilde{y}_1 = \tilde{y}(-1.9) \approx 3.9837 \]

\[ \tilde{y}_2 = \tilde{y}(-1.8) = \tilde{y}_1 + \frac{h x_i^3}{(\tilde{y}_1 + 3)^2} \approx 3.9696 \]

etc.
Error Analysis for FE using
\[
\frac{dy}{dt} = y, \quad y(0) = 1
\]

Exact soln. is \( y = e^t \) with Taylor series expansion \( y = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \ldots \)

<table>
<thead>
<tr>
<th>( t )</th>
<th>Forward Euler</th>
<th>Exact</th>
</tr>
</thead>
<tbody>
<tr>
<td>( t_1 = h )</td>
<td>( y_1 = 1 + h \cdot 1 )</td>
<td>( y = e^h = 1 + h + \frac{h^2}{2!} + \frac{h^3}{3!} + \ldots )</td>
</tr>
<tr>
<td>( t_2 = 2h )</td>
<td>( y_2 = (1+h) + h(1+h) )</td>
<td>( y = e^{2h} = 1 + 2h + \frac{(2h)^2 + (2h)^3}{2!} + \ldots )</td>
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<td></td>
<td>( = (1+h)^2 )</td>
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<td>( = 1 + 2h + h^2 )</td>
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<tr>
<td>( t_3 = 3h )</td>
<td>( y_3 = (1+h)^2 + h(1+h)^2 )</td>
<td>( y = e^{3h} = 1 + 3h + \frac{(3h)^2 + (3h)^3}{2!} + \ldots )</td>
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<td>( = (1+h)^3 )</td>
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<td>( = 1 + 3h + 3h^2 + h^3 )</td>
<td>( = 1 + 3h + \frac{9h^2}{2} + \ldots )</td>
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The "local truncation error" after a single step is:
\[ |y(t_n) - \tilde{y}(t_n)| = ch^2 = O(h^2) \]
is quadratic in \( h \).

More importantly, what is the accumulated error after many steps, or the "global truncation error?"

\[ y = e^{n\lambda} = 1 + nh + \frac{(nh)^2}{2!} + \frac{(nh)^3}{3!} + \cdots \]
\[ = 1 + nh + \frac{n^2h^2}{2} + \cdots \]

\[ \tilde{y} = (1+h)^n = 1 + nh + \frac{n(n-1)}{2} h^2 + \cdots \]

After \( n \) steps
\[ |y - \tilde{y}| = | - \frac{n^2h^2}{2} + \frac{n(n-1)}{2} h^2 + \cdots | \]
\[ = | nh^2 + \cdots | = \ldots \]
with $n = \frac{t_{\text{final}} - t_0}{h}$

then $|y - y'| = \left| \frac{(t_{\text{final}} - t_0)}{h} \frac{h^2}{2} + \ldots \right|$

$= O(h)$

local truncation error is $O(h^2)$

global truncation error is $O(h)$

IF you cut your time step by a factor of 2

$h' = \frac{h}{2}$

then the local error is reduced by 4

but the global error is only reduced by a factor of 2.

FE is explicit, one-step, $O(h)$ accuracy