Review Forward Euler For

\[ \frac{dy}{dt} = f(t, y) \quad y(t_0) = y_0 \]

Make a finite difference approx. \( \frac{dy}{dt} = f(t, y) \)

\[ \text{e.g.} \quad \frac{\hat{y}(t_{n+1}) - \hat{y}(t_n)}{t_{n+1} - t_n} = f(t_n, \hat{y}_n) \]

in shorthand with constant step \( h = t_{n+1} - t_n \):

\[ \hat{y}_{n+1} = \hat{y}_n + h \cdot f(t_n, \hat{y}_n) \quad y(t_0) = y_0 \]

Last lecture consisted of 2 parts:

**Example 1** \( \frac{dy}{dx} = x - y \quad y(0) = 1 \)

Exact Soln. \( y = x - 1 + 2e^{x} - e^{-x} \quad -\infty < x < \infty \)

\( y(0.4) \approx 0.74064 \)
Example 2: Error analysis using
\[ \frac{dy}{dt} = f, \quad y(0) = 1 \]

Exact solution: \[ y = e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots \]

We can compare the exact solution and FE after \( n \) steps at \( t = t_0 + nh = nh \)

Exact: \[ y = e^{nh} = 1 + nh + \frac{(nh)^2}{2!} + \frac{(nh)^3}{3!} + \cdots \]

\[ \tilde{y}(nh) = (1 + h)^n = 1 + nh + \frac{n(n-1)}{2} h^2 + \cdots \]

where \( \cdots \) means higher-order terms (higher powers of \( h \)).
The error is the difference (in absolute value)

\[ |y - \hat{y}| = \left| \frac{1 + nh + \frac{(nh)^2}{2} + \cdots}{1 + n\frac{h}{a} + \frac{n(n-1)h^2}{2a^2} + \cdots} \right| \]

\[ = \left| \frac{nh^2}{a} + \cdots \right| \]

with \( n = \frac{t_{\text{final}} - t_0}{h} \)

After many steps (n large) or

\( t_{\text{final}} - t_0 \gg h \), we arrive at

\[ |y - \hat{y}| \propto h + \text{higher-order terms in } h \]

\[ = \mathcal{O}(h) \]

\[ \Rightarrow \text{decrease the step size by 2, the error decreases by a factor of 2} \]
Now the goal is to construct a scheme, which is more accurate. Let's try for a scheme where the global truncation error is $\propto h^2$.

Then if we decrease the step size by a factor of 2, the global error will be reduced by a factor of 4.

Do this in 2 steps:

1. Backward Euler
2. Modified Euler = Improved Euler = Heun Method = 2nd-order Runge-Kutta

\[
\frac{dy}{dt} = f(t, y) \quad y(t_0) = y_0
\]

\[
\text{BE: } \frac{y_{n+1} - y_n}{\Delta t} = f(t_{n+1}, y_{n+1}) \quad n \geq 0
\]

For constant $h = t_{n+1} - t_n$

\[
y_{n+1} = y_n + h f(t_{n+1}, y_{n+1}) \quad \text{"implicit"}
\]

\[
\text{BE: } \hat{y}_{n+1} = y_n + h f(t_{n+1}, \hat{y}_{n+1})
\]

\[
\text{FE: } \hat{y}_{n+1} = y_n + h f(t_{n+1}, \hat{y}_{n}) \quad \text{"explicit"}
\]
Now the approximate equation is much harder to solve in general!

Our first example is simple enough, so let's try it and compare to FE.

**Example** \( \frac{dy}{dx} = x - y \) \( y(0) = 1 \)

**Exact Soln** \( y = x - 1 + 2e^{-x} \), \(-\infty < x < \infty\)

\[ y(0.4) \approx 0.74064 \]

**FE with** \( h = 0.2 \) \( \Rightarrow \tilde{y}_2(0.4) \approx 0.68 \)

**FE with** \( h = 0.1 \) \( \Rightarrow \tilde{y}_4(0.4) \approx 0.7122 \)

**BE with** \( h = 0.2 \) \( \text{what is } \tilde{y}^*(0.4)? \)

**Step #1**

\( \tilde{y}_1 = y_0 + h(x_1 - \tilde{y}_1) \) \( x_1 = 0.2 \)

\[ \tilde{y}_1 = y_0 + hx_1 - h\tilde{y}_1 \]

\( \tilde{y}_1 + h\tilde{y}_1 = y_0 + hx_1 \)

\( \tilde{y}_1 = \frac{y_0 + hx_1}{1 + h} = \frac{1 + 0.2(0.2)}{1 + 0.2} = 1.04 \approx 0.7667 \)
Step #2 \[ \tilde{y}_2 = \tilde{y}_1 + h(x_2 - y_2) \quad x_2 = 0.4 \]

\[ \tilde{y}_2 = \frac{\tilde{y}_1 + h x_2}{1 + h} = \frac{0.8667 + 0.2(0.4)}{1 + 0.2} = 0.7889 \]

Note that in this particular example, FE gives an underestimate; BE gives an overestimate; this cannot be generalized.

We can repeat for \( h = 0.1 \) 4 steps:

Step #1 \[ x_1 = 0.1 \]

\[ \tilde{y}_1 = \frac{y_0 + h x_1}{1 + h} = 1 + 0.1(0.1) = 0.91818 \]

Step #2 \[ x_2 = 0.2 \]

\[ \tilde{y}_2 = \frac{\tilde{y}_1 + h x_2}{1 + h} = 0.85287 \]

Step #3 \[ \tilde{y}_3 = 0.802627 \]

Step #4 \[ \tilde{y}_4 = \tilde{y}(0.4) \approx 0.76602 \]
Accuracy looks similar, except that FE is an underestimate, BE is an overestimate.

Can we do an error analysis, at least for a simple problem?

### Error analysis for \( y' = y, \ y(0) = 1 \) \ BE

Exact soln. \( y(t) = e^t \)

<table>
<thead>
<tr>
<th>( t )</th>
<th>( \text{BE} )</th>
<th>( \text{Exact} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( t_1 = h )</td>
<td>( \hat{y}_1 = y_0 + h \hat{y}_1 ) ( \hat{y}_1 = \frac{1}{1-h} ) ( = 1 + h + h^2 + h^3 + \cdots )</td>
<td>( y_1 = e^h = 1 + h + \frac{h^2}{2!} + \frac{h^3}{3!} + \cdots )</td>
</tr>
<tr>
<td>( t_2 = 2h )</td>
<td>( \hat{y}_2 = \hat{y}_1 + h \hat{y}_2 ) ( \hat{y}_2 = \frac{1}{1-h} ) ( \cdot ) ( (1-h) \hat{y}_1 ) ( = \frac{1}{(1-h)(1-h)} ) ( = \frac{1}{1-2h} ) ( = 1 + 2h + 3h^2 + \cdots )</td>
<td>( y_2 = e^{2h} ) ( = 1 + 2h + \frac{(2h)^2}{2!} + \frac{(2h)^3}{3!} + \cdots )</td>
</tr>
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</table>

\( = 1 + 2h + 3h^2 + \cdots \)
At each step, the local truncation error is $O(h^2)$.

$\Rightarrow$ the global truncation error is $O(h)$

Rough Argument:

Global error $\propto n \times$ local error

$\propto \left( \frac{t_{\text{final}} - t_0}{h} \right) h^2$

$= O(h)$

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Idea behind Improved Euler is to take an average of the two:

$$\frac{\tilde{y}_{n+1} - \tilde{y}_n}{\tilde{t}_{n+1} - \tilde{t}_n} \approx \left\{ \frac{f(t, \tilde{y}_n) + f(t_{n+1}, \tilde{y}_{n+1})}{2} \right\}$$

but this is implicit and hard to use, so we will massage it a bit to make it explicit.
Here is the hard part

Use Forward Euler for $\tilde{y}_{n+1}$ on the RHS only:

$$
\tilde{y}_{n+1} = \tilde{y}_n + hF(t_n, \tilde{y}_n)
$$

$$
\tilde{u}_{n+1} = \tilde{y}_n + hF(t_n, \tilde{y}_n)
$$

Now this scheme is "explicit", one-step.

Can we demonstrate:

1. Improvement for the specific example

   $$
   \frac{dy}{dx} = x - y \quad y(0) = 1
   $$

2. Global error $\propto h^2$, at least for
   
   a simple example such as $y' = y \quad y(0) = 1$?
Example 1 \[ \frac{dy}{dx} = x - y \quad y(0) = 1 \]

Exact \[ y = x - 1 + 2e^{-x} \quad y(0.4) \approx 0.74664 \]

FE with \( h = 0.2 \) \[ \hat{y}_e(0.4) = 0.68 \]

BE \[ \hat{y}_b(0.4) = 0.79 \]

RK2 with \( h = 0.2 \), 2 steps

Step #1

\[ \hat{y}_1 = y_0 + h \left( \frac{1}{2} [ \hat{y}_0 - y_0 + x_1 - \hat{x}_1 ] \right), \quad \hat{x}_1 = y_0 + h(x_0 - y_0) \]
\[ = y_0 + h \left( \frac{1}{2} [ x_0 - y_0 + x_1 - [y_0 + h(x_0 - y_0)] ] \right) \]
\[ = 1 + \frac{0.2}{2} \left( 0 - 1 + 0.2 - [1 + 0.2(0 - 1)] \right) \]
\[ = 0.84 \]

Step #2

\[ \hat{y}_2 = \hat{y}_1 + h \left( \frac{1}{2} [ \hat{x}_1 - \hat{y}_1 + x_2 - \hat{x}_2 ] \right), \quad \hat{x}_2 = \hat{y}_1 + h(x_1 - \hat{y}_1) \]
\[ = \hat{y}_1 + h \left( \frac{1}{2} [ x_1 - \hat{y}_1 + x_2 - [\hat{y}_1 + h(x_1 - \hat{y}_1)] ] \right) \]
\[ y_{i+1} = y_i + \frac{h}{2} \left[ y_i + \hat{y}_i \right], \quad \hat{y}_i = y_i + hy_0 \]

\[ \tilde{y}_i = y_i + \frac{h}{2} \left( 1 + 1 + h^2 \right) = 1 + h + \frac{h^2}{2} \]

**Exact:** \[ y(h) = e^h = 1 + h + \frac{h^2}{2} + \frac{h^3}{3!} + \cdots \]

**Local error:** \[ O(h^3) \]
\textbf{Step #2}

RK2:
\[
\begin{align*}
\tilde{y}_0 &= \tilde{y}_i + \frac{h}{2} \left( \tilde{y}_i + \tilde{y}_i + \tilde{y}_i \right), \\
\tilde{y}_i &= \tilde{y}_i + \frac{h}{2} \left( \tilde{y}_i + \tilde{y}_i + \tilde{y}_i \right) \\

&= (1 + h + \frac{h^2}{2}) + \frac{h}{2} \left( 2(1 + h + \frac{h^2}{2}) + h \left( 1 + h + \frac{h^2}{2} \right) \right) \\
&= 1 + ah + ah^2 + ah^3 + \frac{ah^4}{4} \\

\text{Exact:} \\
y(2h) &= e^{2h} = 1 + 2h + \frac{(2h)^2}{2!} + \frac{(2h)^3}{3!} + \cdots \\

&= 1 + ah + ah^2 + \frac{ah^3}{6} + \cdots \\

\text{the local error } &\propto h^3 \\

\text{Same rough argument: global error after many steps} \\
&\propto (\# \text{ steps}) h^3 = \left( \frac{t_{\text{final}} - t_0}{h} \right) h^3 \propto h^2 \\

\text{decrease step size by } 2 \Rightarrow \text{global error reduced by a factor of 4}
Example  \( \frac{dy}{dx} = (x+y)^3 \quad y(0) = 1 \)

FE  \( \tilde{y}_{n+1} = \tilde{y}_n + hF(x_n, \tilde{y}_n) \quad x_n = x_0 + nh \)
\[= \tilde{y}_n + h(x_n + \tilde{y}_n)^3 = 2 + nh \]

BE  \( \tilde{y}_{n+1} = \tilde{y}_n + hF(x_n, \tilde{y}_{n+1}) \quad x_n = x_0 + nh \)
\[= \tilde{y}_n + h(x_n + \tilde{y}_{n+1})^3 \]

RK2  \( \tilde{y}_{n+1} = \tilde{y}_n + \frac{h}{2} \left( F(x_n, \tilde{y}_n) + F(x_n, \tilde{y}_{n+1}) \right) \)
\[\tilde{u}_{n+1} = \tilde{y}_n + hF(x_n, \tilde{y}_n) \]
\[\tilde{y}_{n+1} = \tilde{y}_n + \frac{h}{2} \left( (x_n + \tilde{y}_n)^3 \right. \]
\[+ \left( (x_{n+1} + \tilde{y}_n + h(x_n + \tilde{y}_n)^3) \right)^3 \]
Example \( \frac{dy}{dx} = xy^4 \) \( y(-2) = 5 \)

\( x_n = x_0 + nh = -2 + nh \)

FE \( \tilde{y}_{n+1} = \tilde{y}_n + h x_n \tilde{y}_n \)

BE \( \tilde{y}_{n+1} = \tilde{y}_n + h x_{n+1} \tilde{y}_{n+1} \)

RK2 \( \tilde{y}_{n+1} = \tilde{y}_n + \frac{h}{2} \left[ F(x_n, \tilde{y}_n) + F(x_{n+1}, \tilde{u}_{n+1}) \right] \)

\( \tilde{u}_{n+1} = \tilde{y}_n + h F(x_n, \tilde{y}_n) \)

\( \tilde{y}_{n+1} = \tilde{y}_n + \frac{h}{2} \left[ 2 x_n \tilde{y}_n + x_{n+1} \left[ \tilde{y}_n + h x_n \tilde{y}_n \right]^4 \right] \)