Properties of Asymptotic Series

Recall the definition: \( f(x) \sim \sum_{n=0}^{\infty} a_n (x-x_0)^n \), \( x \to x_0 \)

means

\[
\lim_{x \to x_0} \left| f(x) - \sum_{n=0}^{N} a_n (x-x_0)^n \right| \ll |(x-x_0)|^N
\]

as \( x \to x_0 \) for every \( N \)

* For \( f(x) \sim \sum_{n=0}^{\infty} a_n (x-x_0)^n \), \( x \to x_0 \), the \( a_n \)'s are unique.

* But we can add subdominant terms, e.g.,

\[
f(x) + \exp \left[ - (x-x_0)^{-4} \right] \sim \sum_{n=0}^{\infty} a_n (x-x_0)^n \]

will have the same \( a_n \)'s since \( \exp \left[ - (x-x_0)^{-4} \right] \)
decays faster than any power of \( (x-x_0) \)
Consider \( f(x) \sim \sum_{n=0}^{\infty} a_n (x-x_0)^n \), \( x \to x_0 \)
\[ g(x) \sim \sum_{n=0}^{\infty} b_n (x-x_0)^n \], \( x \to x_0 \)

All arithmetic operations are allowed (stated without proof).

1. **Addition:** If \( f(x) \) has an asymptotic series and \( g(x) \) has an asymptotic series as \( x \to x_0 \), then \( \alpha f(x) + \beta g(x) \) has an asymptotic series and we know how to find it:
\[
\alpha f(x) + \beta g(x) \sim \sum_{n=0}^{\infty} (\alpha a_n + \beta b_n) (x-x_0)^n \], \( x \to x_0 \)

2. **Multiplication:** If \( f(x) \) and \( g(x) \) have asymptotic series as \( x \to x_0 \), then \( f(x) g(x) \) has an asymptotic series as \( x \to x_0 \), and we know how to find it:
\[
f(x) g(x) \sim \left( \sum_{n=0}^{\infty} a_n (x-x_0)^n \right) \left( \sum_{n=0}^{\infty} b_n (x-x_0)^n \right) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} a_m b_{n-m} (x-x_0)^n \], \( x \to x_0 \)
as can be checked by direct computation.

\[ f(x) \sim a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots \]

\[ g(x) = b_0 + b_1(x-x_0) + b_2(x-x_0)^2 + \cdots \]

\[ f(x)g(x) \sim \left\{ a_0 + a_1(x-x_0) + \cdots \right\} \left\{ b_0 + b_1(x-x_0) + \cdots \right\} \]

\[ \sim a_0b_0 + \left\{ a_0b_1(x-x_0) + a_1b_0(x-x_0)^2 \right\} \]

\[ + \sum_{n=2}^{\infty} \sum_{m=0}^{n} a_{n-m}b_m(x-x_0)^n \]

\[ + \sum_{n=2}^{\infty} \sum_{m=0}^{n} a_{n-m}b_m(x-x_0)^n (x-x_0)^2 + a_1b_0(x-x_0)^2 + \cdots \]

+ o(x)
3) Division: If \( f(x) \) and \( g(x) \) have asymptotic series as \( x \to x_0 \), then \( \frac{f(x)}{g(x)} \) has an asymptotic series as \( x \to x_0 \).
Proof for multiplication given on pages 125-126 of Bo.

Given:

\[ |f(x) - \sum_{n=0}^{N} a_n (x-x_0)^n| \leq |(x-x_0)^N| \]

\[ |g(x) - \sum_{n=0}^{N} b_n (x-x_0)^n| \leq |(x-x_0)^N| \]

as \( x \to x_0 \) for every \( N \)

Need to show that

\[ |f(x)g(x) - \sum_{n=0}^{N} c_n (x-x_0)^n| \leq |(x-x_0)^N| \]

For every \( N \) as \( x \to x_0 \), where

\[ c_n = \sum_{m=0}^{n} a_m b_{n-m} \]

Basic steps of the proof:

(i) Define \( p = n-m \) and rewrite the difference

\[ f(x)g(x) - \sum_{n=0}^{N} c_n (x-x_0)^n = \]
\[
F(x)g(x) - \sum_{m=0}^{N} a_m (x-x_0)^m \leq \sum_{\rho=0}^{N-m} b_\rho (x-x_0)^\rho \geq \sum_{m=0}^{N} a_m (x-x_0)^m \\
(ii) \text{ add and subtract } g(x) \sum_{m=0}^{N} a_m (x-x_0)^m \\
+ \sum_{m=0}^{N} a_m (x-x_0)^m - \sum_{m=0}^{N} a_m (x-x_0)^m \geq \sum_{\rho=0}^{N-m} b_\rho (x-x_0)^\rho \\
= g(x) \left[ F(x) - \sum_{m=0}^{N} a_m (x-x_0)^m \right] \\
+ \sum_{m=0}^{N} a_m (x-x_0)^m \left[ g(x) - \sum_{\rho=0}^{N-m} b_\rho (x-x_0)^\rho \right] \\
(iii) \text{ Take the limit as } x \to x_0 \text{ and use the given information that } F(x), g(x) \text{ have asymptotic series with coefficients } a_n, b_n \text{ respectively}
Integration is ok but Differentiation is not allowed in general!

Assume that $F(x)$ and $g(x)$ have the same asymptotic series expansion:

$$F(x) \sim g(x) \sim \sum_{n=0}^{\infty} C_n (x-x_0)^n, \quad x \to x_0$$

For example, consider

$$g(x) \sim F(x) + \exp \left[ -\left( x-x_0 \right)^2 \right] \sin \left[ \exp \left[ -\left( x-x_0 \right)^2 \right] \right]$$

* Do $F(x)$ and $g(x)$ have asymptotic expansions?

* Are they the same?
\[ g'(x) \approx f'(x) + 2(x - x_0)^{-3} \exp\left[-(x - x_0)^{-2}\right] \sin^2 \exp\left[(x - x_0)^{-2}\right] \]

The 2nd term oscillates with an amplitude growing like \((x - x_0)^{-3}\) as \(x \to x_0\).

So \(g'(x)\) does not have an asymptotic series of the kind \(g'(x) \sim \sum_{n=0}^{\infty} c_n (x - x_0)^n\) as \(x \to x_0\).

And \(\lim_{x \to x_0} \frac{g'(x)}{f'(x)} \neq 1\)

So \(f'(x)\) is not asymptotic to \(g'(x)\).

* If \(g(x)\) has an asymptotic series, it does not necessarily follow that \(g'(x)\) has an asymptotic series.

* If \(g(x) \sim f(x), x \to x_0\), it does not necessarily follow that \(g'(x) \sim f'(x), x \to x_0\).
But we did assume the existence of derivatives in our ISP analysis. This turns out to be allowed because of the ODE:

\[ y'' + p(x)y' + q(x)y = 0 \]

But \( y(x) = \exp\left[\sum S_i\right] \sim \exp\left[\sum S_0 + S_1 + S_2 + \ldots\right] \]

\( x \to x_0 \)

Can we write?

\[ \sum S_0'' + S_1'' + \ldots \right| + \sum S_0' + S_1' + \ldots \right| \]

\[ + p \sum S_0' + S_1' + \ldots \right| \sim -q \]

\( x \to x_0 \)

We can do this because the ODE guarantees the existence of the derivatives. To see this, consider the simpler case:

\[ y'' = -q(x)y \]

Assume that \( y(x) \) and \( q(x) \) have asymptotic series expansions as \( x \to x_0 \)

Ok to multiply term by term \( \Rightarrow -q(x)y(x) \)

has an asymptotic series (AS)

Ok to integrate term by term \( \Rightarrow \)
\[ y' = -\int_{x_0}^{x} g(s) y(s) \, ds + C, \quad x \to x_0 \]

So both \( y'' = -qy \) and \( y' = -\int_{x_0}^{x} g(s) y(s) \, ds + C \)

have AS as \( x \to x_0 \) and our steps are justified

The more general case \( y'' + p(x)y' + q(x)y = 0 \)

Assume that \( p(x), q(x), y(x) \) have AS

we can form an integrating factor

\[ \mu(x) = \exp \left[ \int p(x) \, dx \right] \] since integration is allowed

Then \( ** \) becomes

\[ \frac{d}{dx} \left[ \mu(x) y'(x) \right] = -\mu(x) q(x) y(x) \]

RHS OK
integration ok \Rightarrow
\[ \mu(x) y'(x) = - \int \mu(s) q(s) y(s) \, ds + C \]
RHS ok

division ok \Rightarrow
\[ y'(x) = - \frac{1}{\mu(x)} \int \mu(s) q(s) y(s) \, ds + \frac{c}{\mu(x)} \]
RHS ok

\[ \Rightarrow \ y'(x) \text{ has AS} \]

\[ \Rightarrow \ y'' = - py' - qy \text{ has AS} \]