Math 322 Lecture 8

Moving on to 2D; Lets consider steady-state with non-zero Dirichlet boundary conditions

$\Longrightarrow$ Laplace's Eqn. in 2D

$\boxed{\#1}$ Cartesian $\boxed{\#2}$ Polar

$\boxed{\#1}$ $\dfrac{\partial u}{\partial t} = k\left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}\right)$ $\quad u = u(x,y,t)$

$0 < x < L, \quad 0 < y < H, \quad t > 0$

Boundary conditions:

$u(x=0, y, t) = g_1(y) \qquad u(L, y, t) = g_2(y)$

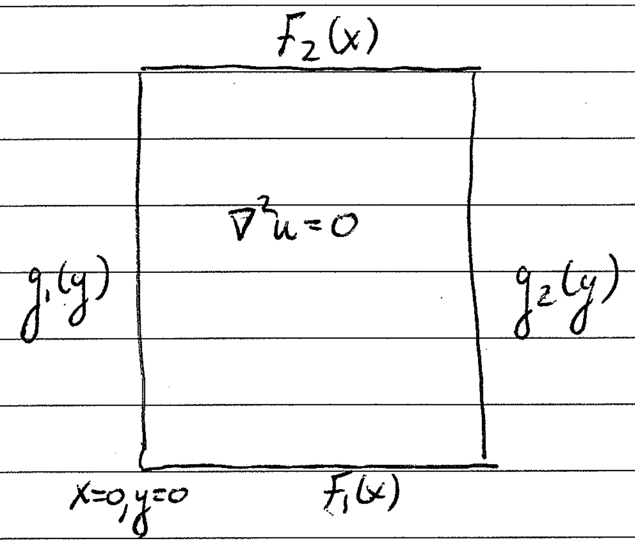
$u(x, y=0, t) = f_1(x) \qquad u(x, H, t) = f_2(x)$

With b.c.s independent of time, we can expect a steady state solution, and with Dirichlet boundary conditions, we also anticipate that the initial conditions will not matter (recall the 1D problem we solved).

Lets look for a steady-state solution with $\frac{\partial u}{\partial t}(x,y,t) = 0$

$$\Rightarrow \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad u = u(x,y) \qquad \begin{matrix} 0 < x < L \\ 0 < y < H \end{matrix}$$

$$u(0,y) = g_1(y) \quad u(L,y) = g_2(y) \quad u(x,0) = f_1(x) \quad u(x,H) = f_2(x)$$



Strategy: Since the PDE is linear, homogeneous $\Rightarrow$ we can use the Principle of Superposition

Try to add up solutions to simpler problems, all satisfying the same PDE in the interior, but different boundary conditions that also "add up."

$$u(x,y) = u_1(x,y) + u_2(x,y) + u_3(x,y) + u_4(x,y)$$

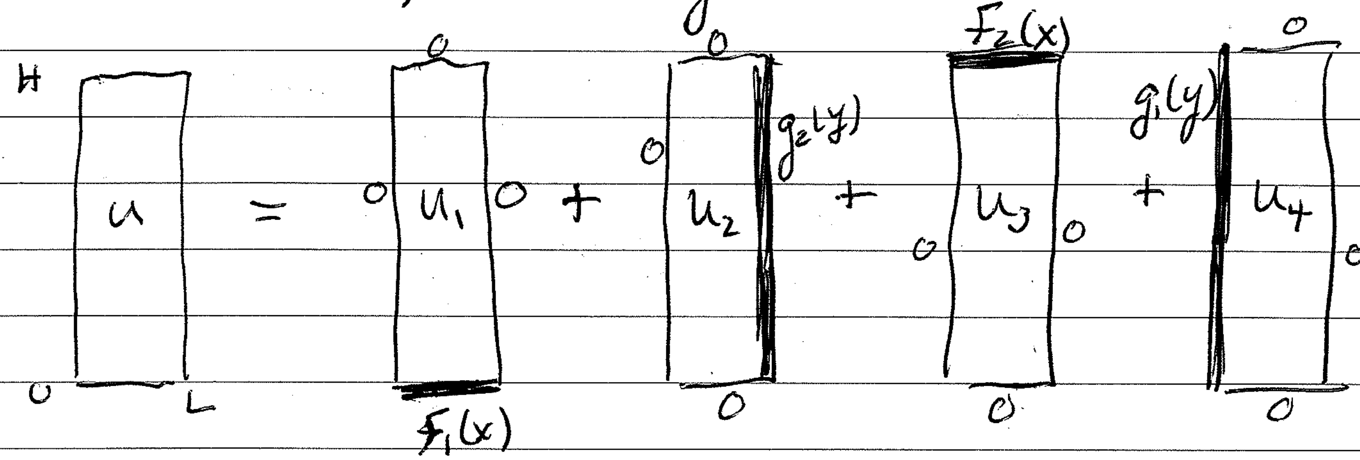
$$\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial y^2} = 0 \qquad 0 < x < L, \quad 0 < x < H$$

$$\frac{\partial^2 u_2}{\partial x^2} + \frac{\partial^2 u_2}{\partial y^2} = 0 \qquad 0 < x < L, \quad 0 < x < H$$

etc.

such that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0 \qquad 0<x<L,\ 0<y<H$

Now let the b.c.s for each problem be homogeneous on 3 sides, non-homogeneous on 1 side



$$u(0,y) = u_1(0,y) + u_2(0,y) + u_3(0,y) + u_4(0,y) = g_1(y)$$

$$u(L,y) = u_1(L,y) + u_2(L,y) + u_3(L,y) + u_4(L,y) = g_2(y)$$

etc.

Therefore $u(x,y)$ satisfies the original boundary conditions

Now we will see that we can fall back on our eigenvalue / eigenfunction method for sub-problems #1, #2, #3, #4

Book solves #4

#3

$$\frac{\partial^2 u_3}{\partial x^2} + \frac{\partial^2 u_3}{\partial y^2} = 0 \qquad 0 < x < L, \quad 0 < y < H$$

$u_3(0,y) = 0 \quad u_3(L,y) = 0 \quad u_3(x,0) = 0 \quad u_3(x,H) = F_2(x)$

Eqn is linear homogeneous $\Rightarrow$ Try $u_3(x,y) = \phi(x)G(y)$

Plug in: $\frac{d^2\phi(x)}{dx^2} G(y) + \phi(x)\frac{d^2G(y)}{dy^2} = 0$

Divide by $\phi(x)G(y)$ {formal} $\Rightarrow$

$$\frac{1}{\phi(x)}\frac{d^2\phi(x)}{dx^2} = \frac{-1}{G(y)}\frac{d^2G(y)}{dy^2} = \pm\lambda$$

$\uparrow$ which is more convenient?

Lets look at the boundary conditions and try to fall back on a problem we've solved before

$u_3(0,y) = 0 \quad \Rightarrow \quad \phi(0) = 0$

$u_3(L,y) = 0 \quad \Rightarrow \quad \phi(L) = 0$

$u_3(x,0) = 0 \quad \Rightarrow \quad G(0) = 0$

$u_3(x,H) = F_2(x) \quad \Rightarrow \quad \phi(x)\,G(H) = F_2(x)$

(a) $\frac{1}{\phi(x)} \frac{d^2\phi(x)}{dx^2} \overset{?}{=} \pm \lambda \qquad \phi(0) = \phi(L) = 0$

Looks like an old problem if we choose − sign.

Then we know $\lambda > 0$

$$\frac{d^2\phi}{dx^2} = -\lambda\,\phi(x) \qquad \phi(0) = \phi(L) = 0$$

$$\lambda_n = \frac{n^2\pi^2}{L^2} \quad n = 1, 2, 3, \ldots \; ; \quad \phi_n(x) = b_n \sin\frac{n\pi x}{L}$$

(b) $\frac{d^2G(y)}{dy^2} = +\lambda\,G(y) \qquad$ Try $G(y) = Ce^{ry}$

$$\Rightarrow r^2 Ce^{ry} - \lambda Ce^{ry} = 0 \quad \Rightarrow r^2 = \lambda$$

$$r_{1,2} = \pm\sqrt{\lambda}$$

For each $\lambda_n = \frac{n^2\pi^3}{L^2}$:

$$G_n(y) = C_1 \sinh\left(\sqrt{\lambda_n}\, y\right) + C_2 \cosh\left(\sqrt{\lambda_n}\, y\right)$$

and we must enforce $G_n(0) = 0 \Rightarrow C_2 = 0$

$$(u_3)_n(x,y) = \phi_n(x)\, G_n(y)$$

$$= B_n \sin\frac{n\pi x}{L} \sinh\frac{n\pi y}{L}$$

$$u_3(x,y) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} \sinh\frac{n\pi y}{L}$$

How do we enforce $u_3(x, H) = f_2(x)$?

$$\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} \sinh\frac{n\pi H}{L} = f_2(x)$$

Now use orthogonality

$$\int_0^L \sin\frac{m\pi x}{L} \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} \sinh\frac{n\pi H}{L}\, dx$$

$$= \int_0^L \sin\frac{m\pi x}{L} f_2(x)\, dx$$

Note $\sinh\frac{n\pi H}{L}$ does not depend on x

$$\int_0^L \sum_{n=1}^{\infty} B_n \sinh\frac{n\pi H}{L} \sin\frac{n\pi x}{L} \sin\frac{m\pi x}{L}\, dx$$

$$= \int_0^L \sin\frac{m\pi x}{L} f_2(x)\, dx$$

Assume we can interchange integration / summation

$$\sum_{n=1}^{\infty} B_n \sinh\frac{n\pi H}{L} \int_0^L \sin\frac{n\pi x}{L} \sin\frac{m\pi x}{L}\, dx$$

$$= \int_0^L \sin\frac{m\pi x}{L} f_2(x)\, dx$$

only $n=m$ on LHS gives a non-zero integral:

$$B_m \sinh\frac{m\pi H}{L} \frac{L}{2} = \int_0^L \sin\frac{m\pi x}{L} f_2(x)\, dx$$

$$\Rightarrow \quad B_m = \frac{2}{L} \frac{1}{\sinh\frac{m\pi H}{L}} \int_0^L \sin\frac{m\pi x}{L} f_2(x)\, dx$$

$$\boxed{\begin{aligned} u_3(x,y) &= \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} \sinh\frac{n\pi y}{H} \\ B_n &= \frac{2}{L} \frac{1}{\sinh\frac{n\pi H}{L}} \int_0^L \sin\frac{n\pi x}{L} f_2(x)\, dx \end{aligned}}$$

$\boxed{\# 4}$ Summary $\frac{\partial^2 u_4}{\partial x^2} + \frac{\partial^2 u_4}{\partial y^2} = 0 \quad \begin{matrix} 0<x<L \\ 0<y<H \end{matrix}$

$u_4(0,y) = g_1(y)$, $u_4(L,y) = 0$, $u_4(x,0) = 0$, $u_4(x,H) = 0$

Let $u_4(x,y) = R(x)S(y) \Rightarrow$

$$\frac{1}{R(x)} \frac{d^2R(x)}{dx^2} = \frac{-1}{S(y)} \frac{d^2S(y)}{dy^2} = \pm \lambda$$

$\uparrow$ for convenience ?

y-dependence $\quad \frac{-1}{S(y)} \frac{d^2S(y)}{dy^2} = +\lambda$

$S(0) = S(H) = 0$ + sign is more convenient

$\Rightarrow \quad \lambda_n = \frac{n^2\pi^2}{H^2} \quad n = 1, 2, 3 \ldots$

$S_n(y) = d_n \sin\frac{n\pi y}{H}$

x-dependence $\quad \frac{d^2R(x)}{dx^2} = \lambda R(x)$

with $R(x) = C_1 e^{\sqrt{\lambda}x} + C_2 e^{-\sqrt{\lambda}x}$ or ...

$R(x) = C_7 \sinh(\sqrt{\lambda}(x-L)) + C_8 \cosh(\sqrt{\lambda}(x-L))$

then easy to apply the b.c. at $x = L$

$R(L) = 0 \Rightarrow C_8 = 0$

Details

$\boxed{\# 4}$ $$\frac{\partial^2 u_4}{\partial x^2} + \frac{\partial^2 u_4}{\partial y^2} = 0 \qquad \begin{array}{l} 0 < x < L \\ 0 < y < H \end{array}$$

$$u_4(0,y) = g_1(y) \quad u_4(L,y) = 0 \quad u_4(x,0) = 0 \quad u_4(x,H) = 0$$

Let $u_4(x,y) = R(x)S(y) \implies$

$$\frac{1}{R(x)}\frac{d^2R(x)}{dx^2} = \frac{-1}{S(y)}\frac{d^2S(y)}{dy^2} = \underset{?}{\pm}\,\lambda$$

which is convenient

B.C.s :

$$R(0)S(y) = g_1(y) \quad R(L)S(y) = 0 \quad R(x)S(0) = 0 \quad R(x)S(H) = 0$$

y-dependence: $$-\frac{1}{S(y)}\frac{d^2S(y)}{dy^2} = \underset{?}{\pm}\,\lambda$$

with $S(0) = S(H) = 0$ suggest + sign

$$\implies \frac{d^2S(y)}{dy^2} = -\lambda S(y) \qquad S(0) = S(H) = 0$$

such that $\lambda_n = \frac{n^2\pi^2}{H^2}$ $n = 1, 2, 3, \dots$

$$S_n(y) = d_n \sin\frac{n\pi y}{H}$$

x-dependence $\frac{d^2R(x)}{dx^2} = \lambda R(x)$

with $R(x) = C_1 e^{\sqrt{\lambda} x} + C_2 e^{-\sqrt{\lambda} x}$

or $R(x) = C_3 \sinh\sqrt{\lambda} x + C_4 \cosh\sqrt{\lambda} x$

or $R(x) = C_5 e^{\sqrt{\lambda} x} e^{-\sqrt{\lambda} L} + C_6 e^{-\sqrt{\lambda} x} e^{\sqrt{\lambda} L}$

$= C_5 e^{\sqrt{\lambda}(x-L)} + C_6 e^{-\sqrt{\lambda}(x-L)}$

or $R(x) = C_7 \sinh(\sqrt{\lambda}(x-L)) + C_8 \cosh(\sqrt{\lambda}(x-L))$

The latter turns out to be more convenient ...
simple to apply the condition at $x=L$

$R(L) = 0 \Rightarrow C_8 = 0 \Rightarrow R(x) = C_7 \sinh(\sqrt{\lambda}(x-L))$

So for each $\lambda_n = \frac{n^2\pi^2}{H^2}$ $n=1,2,3\ldots$

$R_n(x) = C_n \sinh\left(\frac{n\pi}{H}(x-L)\right)$

For each λ_n:

$u_4(x,y) = D_n \sin\frac{n\pi y}{H} \sinh\left(\frac{n\pi}{H}(x-L)\right)$

By superposition :

$$u_4(x,y) = \sum_{n=1}^{\infty} D_n \sin\frac{n\pi y}{H} \sinh\left(\frac{n\pi}{H}(x-L)\right)$$

Now apply non-homogeneous condition

$$u_4(0,y) = g_1(y) = \sum_{n=1}^{\infty} D_n \sin\frac{n\pi y}{H} \sinh\left(\frac{-n\pi L}{H}\right)$$

To find D_n's, use orthogonality :

$$\int_0^H \sin\frac{m\pi y}{H} \sum_{n=1}^{\infty} D_n \sin\frac{n\pi y}{H} \sinh\left(-\frac{n\pi L}{H}\right) dy$$

$$= \int_0^H \sin\frac{m\pi y}{H}\, g_1(y)\, dy$$

$$\int_0^H \sum_{n=1}^{\infty} D_n \sinh\left(\frac{-n\pi L}{H}\right) \sin\frac{m\pi y}{H} \sin\frac{n\pi y}{H}\, dy$$

$$= \int_0^H \sin\frac{m\pi y}{H}\, g_1(y) dy$$

interchange order of summation / integration

$$\sum_{n=1}^{\infty} D_n \sinh\left(\frac{-n\pi L}{H}\right) \int_0^H \sin\frac{m\pi y}{H} \sin\frac{n\pi y}{H}\, dy$$

$$= \int_0^H \sin\frac{m\pi y}{H}\, g_1(y)\, dy$$

On LHS, only $n=m$ is non zero $\Rightarrow$

$$D_m \sinh\left(\frac{-m\pi L}{H}\right) \frac{H}{2} = \int_0^H \sin\frac{m\pi y}{H}\, g_1(y)\, dy$$

or $$D_m = \frac{2}{H} \frac{1}{\sinh\left(\frac{-m\pi L}{H}\right)} \int_0^H \sin\frac{m\pi y}{H}\, g_1(y)\, dy$$

$$\boxed{\begin{aligned} u_4(x,y) &= \sum_{n=1}^{\infty} D_n \sin\frac{n\pi y}{H} \sinh\left(\frac{n\pi}{H}(x-L)\right) \\ D_n &= \frac{2}{H} \frac{1}{\sinh\left(\frac{-m\pi L}{H}\right)} \int_0^H \sin\frac{m\pi y}{H}\, g_1(y)\, dy \end{aligned}}$$

$$u(x,y) = u_1(x,y) + u_2(x,y) + u_3(x,y) + u_4(x,y)$$