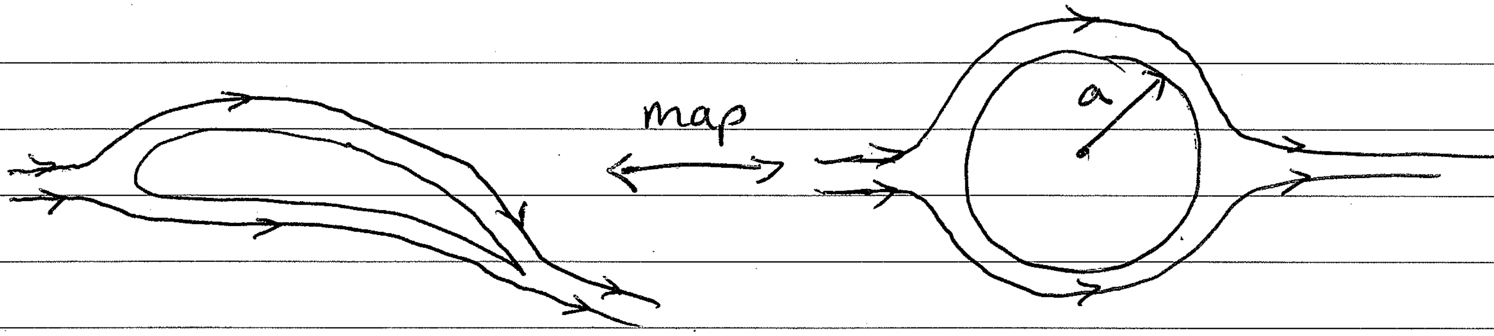
Math 322 Lecture 9

Laplace's Eqn. in a circular disk or outside a circular disk. Inside important for theory.

The problem outside a circular disk is important in the history of aerodynamics! There is a simple map that take the airfoil shape to a circle.



So one can map the flow around a 2D airfoil to flow around the 2D circular cylinder, and then obtain formulas for Lift and Drag (=0)

Laplace's Eqn. in Circular Polar Coordinates:

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 \qquad \begin{matrix} 0 \le r < a \\ -\pi < \theta \le \pi \end{matrix}$$

with boundary condition $u(a,\theta) = f(\theta)$

$r=a$ is the radius of the circle

Inside the circle we have boundedness at $r=0$:

$|u(0,\theta)| < \infty$

We can impose periodicity in θ $\Rightarrow$

$u(r,-\pi) = u(r,\pi)$

$\frac{\partial u}{\partial \theta}(r,-\pi) = \frac{\partial u}{\partial \theta}(r,\pi)$

PDE is linear, homogeneous, so try

$u(r,\theta) = \phi(\theta)\,G(r) \Rightarrow$

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dG}{dr}\right)\phi(\theta) + \frac{1}{r^2}\,G(r)\frac{d^2\phi(\theta)}{d\theta^2} = 0$$

Now multiply by $\frac{r^2}{G(r)\,\phi(\theta)}$: (Formal)

$$\frac{r}{G(r)}\frac{d}{dr}\left(r\frac{dG(r)}{dr}\right) = -\frac{1}{\phi(\theta)}\frac{d^2\phi(\theta)}{d\theta^2} = \lambda$$

Choose $+\lambda$ because we already know how to solve

$$\frac{d^2\phi(\theta)}{d\theta^2} = -\lambda\phi(\theta) \qquad \phi(-\pi) = \phi(\pi)$$
$$\frac{d\phi}{d\theta}(-\pi) = \frac{d\phi}{d\theta}(\pi)$$

$\lambda = 0 \quad \phi(\theta) = a_0$

$\lambda = \frac{n^2\pi^2}{\pi^2} \quad n = 1, 2, 3 \ldots \qquad (L = \pi)$

$$\phi(\theta) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\theta + \sum_{n=1}^{\infty} b_n \sin n\theta$$

Now the problem for r-dependence:

$$\frac{r}{G(r)}\frac{d}{dr}\left(r\frac{dG(r)}{dr}\right) = \lambda \quad \text{(zero or positive)}$$

$$\frac{r}{G}\frac{dG}{dr} + \frac{r^2}{G}\frac{d^2G}{dr^2} = \lambda \quad \text{(chain rule)}$$

$$r^2G'' + rG' - \lambda G = 0 \qquad \lambda = 0 \text{ or } n^2$$
$$n = 1, 2, 3 \ldots$$

This is a Cauchy-Euler or <u>Equidimensional</u> Equation

Why?

Its solutions are powers of r

Trial solution $G(r) = A r^p \Rightarrow$

$G'(r) = A p r^{p-1}$, $G''(r) = A p(p-1) r^{p-2}$

Plug in:

$$A r^p \left[p(p-1) + p - n^2 \right] = 0$$

$$\Rightarrow \quad p(p-1) + p - n^2 = 0$$

$$\Rightarrow \quad \left.\begin{array}{ll} p^2 = 0 & \text{for } \lambda = 0 \\ p^2 = n^2 & \text{for } \lambda = n^2 \end{array}\right\}$$

For the case $\lambda = n^2$, we have $p = \pm n$

$$G_n(r) = C_{1n} r^n + C_{2n} r^{-n}$$

For the case $\lambda = 0$, we have

$$G_0(r) = C_0$$

and our trial solution only generated one linearly independent solution. We need a second linearly independent solution in this case!

In general, we could use the Method of Reduction of order ...

but here its simpler to return to the original equation

$$\frac{r}{G}\frac{d}{dr}\left(r\frac{dG}{dr}\right) = \lambda = 0$$

$$\Rightarrow \quad \frac{d}{dr}\left(r\frac{dG(r)}{dr}\right) = 0 \quad \text{integrate once}$$

$$\Rightarrow \quad r\frac{dG(r)}{dr} = A \quad \Rightarrow \quad \frac{dG(r)}{dr} = \frac{A}{r}$$

integrate again $\Rightarrow$ $G(r) = A\ln r + B$

{ 2 linearly independent solutions : (i) constant, (ii) $\ln r$

$$G_n(r) = A\ln r + B \qquad n=0$$

$$G_n(r) = C_{1n}r^n + C_{2n}r^{-n} \qquad n=1,2,3\ldots$$

Recall our conditions for the r-dependence

$u(a,\theta) = F(\theta) \qquad |u(0,\theta)| < \infty$

and from the boundedness condition we set $A = 0$, $C_{2n} = 0$

$$G(r) = B + \sum_{n=1}^{\infty} C_{1n} r^n$$

$$u_n(r,\theta) = \phi_n(\theta)\, G_n(r)$$

$$= a_0 B + C_{1n}\left(a_n \cos n\theta + b_n \sin n\theta\right) r^n$$

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty}\left(A_n \cos n\theta + B_n \sin n\theta\right) r^n$$

$$u(a,\theta) = F(\theta)$$

Use orthogonality to find A_0, A_n, B_n

$$u(a,\theta) = F(\theta) = A_0 + \sum_{n=1}^{\infty}\left(A_n \cos n\theta + B_n \sin n\theta\right) a^n$$

e.g. multiply by $\cos m\theta$ and integrate $[-\pi, \pi]$

$m = 0$ $$A_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(\theta)\, d\theta$$

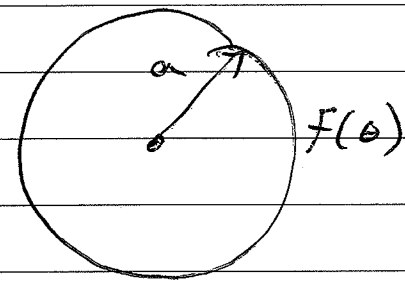
$m = 1, 2, 3 \ldots$ $$A_m = \frac{1}{a^m \pi} \int_{-\pi}^{\pi} F(\theta) \cos m\theta \, d\theta$$

multiply by $\sin m\theta$, integrate

$$B_m = \frac{1}{a^m \pi} \int_{-\pi}^{\pi} F(\theta) \sin m\theta \, d\theta$$

Qualitative Properties of Laplace's Eqn.

Mean Value Thm



Inside the circle :

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} \{A_n \cos n\theta + B_n \sin n\theta\} r^n$$

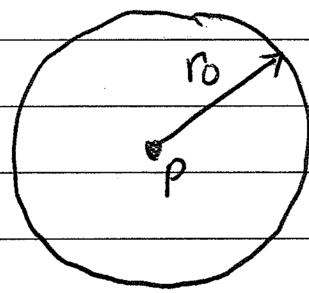
At the origin $r=0$

$$u(0,\theta) = A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(\theta)\,d\theta$$

the mean of the boundary values

Maximum Principle (Minimum Principle)

Assuming no sources/sinks inside and steady state the temp cannot attain its max (min) inside the domain



$$u(P,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\, d\theta$$

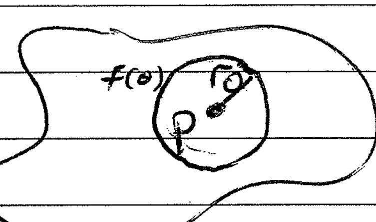
identify the max / min values of $f(\theta)$:
f_{max} and f_{min} (constants)

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f_{min}\, d\theta \le u(P,\theta) \le \frac{1}{2\pi}\int_{-\pi}^{\pi} f_{max}\, d\theta$$

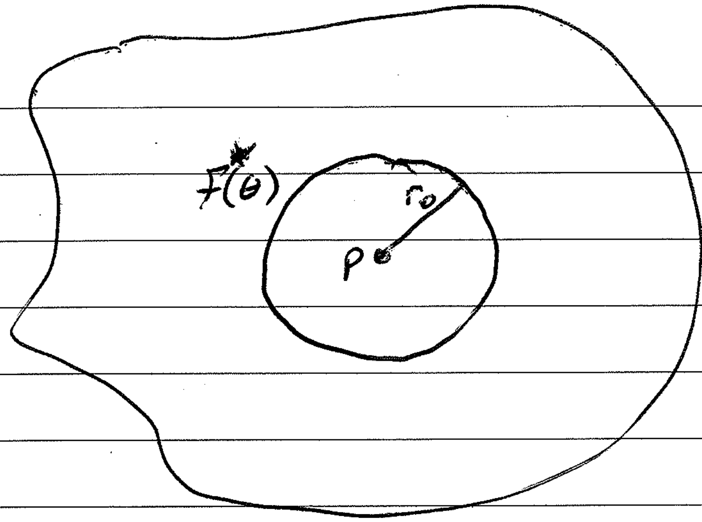
$$\frac{2\pi}{2\pi} f_{min} \le u(P,\theta) \le \frac{2\pi}{2\pi} f_{max}$$

$$f_{min} \le u(P,\theta) \le f_{max}$$

Now generalize to any region R



see pp. 79
Proof by contradiction



Proof by Contradiction:

Assume $u(x,y)$ attains its max or min at P

The value of $u(x,y)$ at P is the average of the $f^*(\theta)$ [the function of θ on the circle centered at point P with radius r_0

These statement contradict each other [the average cannot be a max or min]