MATH320 Homework 1 Solutions
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Section 1.2 30.
v(t): Speed of car at time t(sec)
s(t): Traveling distance of car until time t(sec)
t = 0: The moment when the brakes are applied
t = T_{\text{stop}}: The moment when the car completely stop

\[ S(T_{\text{stop}}) = 176 \text{ ft} \quad v(0) = 88 \text{ft/s} \]

Step 1. Differential equations of \( v \) and \( s \):

\[ \frac{dv}{dt} = -k \quad (k > 0 : \text{constant}) \implies v(t) = -kt + C \]

\[ v(0) = 88 \implies v(t) = -kt + 88 \]

\[ \frac{ds}{dt} = v(t) \implies s(t) = \int_0^t v(z) dz = \int_0^t (-kz + 88) dz = -\frac{1}{2}kt^2 + 88t \]

Step 2.

\[ v(T_{\text{stop}}) = -kT_{\text{stop}} + 88 = 0 \implies T_{\text{stop}} = \frac{88}{k} \]

\[ s(T_{\text{stop}}) = -\frac{1}{2}kT_{\text{stop}}^2 + 88T_{\text{stop}} = 176 \quad \implies \quad \frac{1}{2}T_{\text{stop}}^2 = 176 - \frac{88}{k}T_{\text{stop}} = 176 \]

Therefore,

\[ 44T_{\text{stop}} = 176 \iff T_{\text{stop}} = 4 \text{ seconds} \]

And the deceleration is

\[ -22 \text{ ft/s}^2 \]

Section 1.3 30.

\[ y = \begin{cases} 
1 & \text{if } x \leq c \\
\cos(x - c) & \text{if } c < x < c + \pi \\
-1 & \text{if } x \geq c + \pi 
\end{cases} \]

(1) First of all, we show that \( y \) defined as above is a solution for \( y' = -\sqrt{1 - y^2} \) for all \( x \).

Case 1. If \( x \neq c \) and \( x \neq c + \pi \)

then,

\[ y' = \begin{cases} 
0 & \text{if } x < c \\
-\sin(x - c) & \text{if } c < x < c + \pi \\
0 & \text{if } x > c + \pi 
\end{cases} \]

\[ -\sqrt{1 - y^2} = \begin{cases} 
0 & \text{if } x < c \\
-\sqrt{1 - \cos^2(x - c)} = -\sqrt{\sin^2(x - c)} = -|\sin(x - c)| & \text{if } c < x < c + \pi \\
0 & \text{if } x > c + \pi 
\end{cases} \]
Thus,

\[ y' = -\sqrt{1 - y^2} \]

if \( x \neq c, c + \pi \).

Case 2. If \( x = c \) or \( c + \pi \), then,

\[
\lim_{{x \to c^-}} y'(x) = 0 \\
\lim_{{x \to c^+}} y'(x) = -\lim_{{x \to c^+}} \sin(x - c) = \sin 0 = 0 \\
\Rightarrow \lim_{{x \to c^-}} y'(x) = \lim_{{x \to c^+}} y'(x)
\]

So,

\[ y'(c) = 0 \]

Thus,

\[ y'(c) = 0 = -\sqrt{1 - y^2(x)} \]

One can argue similarly at \( x = c + \pi \) as well.

(b) Let us consider the differential equation

\[ y' = -\sqrt{1 - y^2} \]

with the initial condition

\[ y(a) = b. \]

For any given \( a \),

(i) If \( |b| > 1 \), then \(-\sqrt{1 - y(a)} = -\sqrt{1 - b^2} \) is not a real number. So there is no solution of the differential equation above. (ii) If \( |b| < 1 \), then \(-\sqrt{1 - y^2} \) and \( \frac{\partial \sqrt{1 - y^2}}{\partial y} = \frac{y}{\sqrt{1 - y^2}} \) are continuous for \((x, y)\) near the point \((a, b)\). Thus, there exists unique solution of the differential equation with the initial condition \( y(a) = b \).

(iii) If \( |b| = 1 \), there are infinitely many solution of the differential equation. Because the function

\[ y = \begin{cases} 
1 & \text{if } x \leq c \\
\cos(x - c) & \text{if } c < x < c + \pi \\
-1 & \text{if } x \geq c + \pi
\end{cases} \]

are solutions if we choose \( c \) appropriately. There are infinitely many possible choices of \( c \) for that.

(Example) Consider the problem

\[ y' = -\sqrt{1 - y^2}, \quad y(0) = 1. \]

Then, for all \( c \leq 0 \), the function \( y = \begin{cases} 
1 & \text{if } x \leq c \\
\cos(x - c) & \text{if } c < x < c + \pi \\
-1 & \text{if } x \geq c + \pi
\end{cases} \) is a solution of the problem.
The differential equation
\[(x^2 + 1)(\tan y) y' = x\]
is separable, so we proceed accordingly. We have
\[
\int \tan y \, dy = \int \frac{x}{x^2 + 1} \, dx.
\]
(Note that there is no risk in dividing by \(x^2 + 1\), which is never zero.) Integrating, we obtain
\[
\log |\sec y| = \frac{1}{2} \log(x^2 + 1) + C,
\]
where \(C\) is a constant of integration. (For the integral on the left, write tangent as sine over cosine and make the substitution \(v = \cos y\). For the one on the right, make the substitution \(u = x^2 + 1\).) Then
\[
|\sec y| = e^{\frac{1}{2} \log(x^2+1)+C} = e^C \cdot e^{\frac{1}{2} \log(x^2+1)} = e^C \cdot e^{\log(x^2+1)^{1/2}} = e^C (x^2 + 1)^{1/2} = D(x^2 + 1)^{1/2},
\]
where \(D = e^C\) is a (positive) constant. The presence of the absolute value around \(\sec y\) tells us that a general solution curve is not the graph of a function. We can leave our answer as
\[
|\sec y| = D(x^2 + 1)^{1/2}
\]
or, wishing to show some style, we can square both sides of this equation (without introducing extraneous solutions) and write
\[
(\sec y)^2 = E(x^2 + 1),
\]
where \(E = D^2\) is yet another constant. The answer in the book is not quite correct.
Let $P(t)$ denote the number of potassium atoms in the moon rock $t$ years from the time when it contained only potassium. We know that

$$P(t) = P_0 e^{-kt},$$

where $P_0$ is the number of potassium atoms at $t = 0$. (Please consult the textbook for the derivation of this equation.) The constant $k$ is related to the half-life $\tau$ of potassium by the equation

$$k = \frac{\log 2}{\tau}.$$

(Again, please consult the textbook.) Let $A(t)$ be the number of argon atoms in the rock. Since, on average, one argon atom is created after nine potassium atoms have disintegrated, we have

$$A(t) = \frac{P_0 - P(t)}{9}.$$

(Note that $P_0 - P(t)$ is the number of potassium atoms that have disintegrated. Also, observe that $A(0) = 0$, as expected.) At present, there are equal numbers of potassium and argon atoms in the rock. Suppose that this is $T$ years from the time when the rock contained only potassium. Then

$$P(T) = A(T) = \frac{P_0 - P(T)}{9},$$

so

$$P(T) = \frac{1}{10} P_0.$$

Consequently,

$$P_0 e^{-kT} = \frac{1}{10} P_0.$$

Solving this equation, we obtain

$$T = \frac{\log 10}{k} = \frac{\log 10}{\log 2} \tau = \frac{\log 10}{\log 2} (1.28 \times 10^9 \text{ yr}) \approx 4.25 \times 10^9 \text{ yr}.$$

The answer in the book is incorrect, again.