MATH320 Homework 1 Solutions

Written by Myoungjean Bae and Anakewit(Tete) Boonkasame

Section 1.2 30.
$v(t)$: Speed of car at time t(sec)
$s(t)$: Traveling distance of car until time t(sec)
$t = 0$: The moment when the brakes are applied
$t = T_{stop}$: The moment when the car completely stop

$$S(T_{stop}) = 176 \;\; ft \quad v(0) = 88 ft/s$$

Step1. Differential equations of v and s:

$$\frac{dv}{dt} = -k \;\; (k > 0 : \text{constant}) \Longrightarrow v(t) = -kt + C \overset{v(0)=88}{\Longrightarrow} v(t) = -kt + 88$$
$$\frac{ds}{dt} = v(t) \Longrightarrow s(t) = \int_0^t v(z)dz = \int_0^t -kz + 88dz = -\frac{1}{2}kt^2 + 88t$$

Step2.

$$v(T_{stop}) = -kT_{stop} + 88 = 0 \Longrightarrow T_{stop} = \frac{88}{k}$$
$$s(T_{stop}) = -\frac{1}{2}kT_{stop}^2 + 88T_{stop} = 176 \overset{k=\frac{88}{T_{stop}}}{\Longrightarrow} -\frac{1}{2}\frac{88}{T_{stop}}T_{stop}^2 + 88T_{stop} = 176$$

Therefore,

$$44T_{stop} = 176 \Longleftrightarrow T_{stop} = 4 \text{ seconds}$$

And the deceleration is

$$-22 \; ft/s^2$$

Section 1.3 30.

$$y = \begin{cases} 1 & \text{if } x \le c \\ \cos(x-c) & \text{if } c < x < c+\pi \\ -1 & \text{if } x \ge c+\pi \end{cases}$$

(1) First of all, we show that y defined as above is a solution for $y' = -\sqrt{1-y^2}$ for <u>all</u> x.
Case 1. If $x \ne c$ and $x \ne c + \pi$
then,

$$y' = \begin{cases} 0 & \text{if } x < c \\ -\sin(x-c) & \text{if } c < x < c+\pi \\ 0 & ; \text{if } x > c+\pi \end{cases}$$

$$-\sqrt{1-y^2} = \begin{cases} 0 & \text{if } x < c \\ -\sqrt{1-\cos^2(x-c)} = -\sqrt{\sin^2(x-c)} = -|\sin(x-c)| \overset{0<x-c<\pi}{=} -\sin(x-c) & \text{if } c < x < c+\pi \\ 0 & ; \text{if } x > c+\pi \end{cases}$$

Thus,

$$y' = -\sqrt{1-y^2}$$

if $x \neq c, c+\pi$.
Case 2. If $x = c$ or $c+\pi$, then,

$$\lim_{x\to c-} y'(x) = 0$$
$$\lim_{x\to c+} y'(x) = -\lim_{x\to c+} \sin(x-c) = \sin 0 = 0$$
$$\Longrightarrow \lim_{x\to c-} y'(x) = \lim_{x\to c+} y'(x)$$

So,

$$y'(c) = 0$$

Thus,

$$y'(c) = 0 = -\sqrt{1-y^2(x)}$$

One can argue similarly at $x = c+\pi$ as well.
(b) Let us consider the differential equation

$$y' = -\sqrt{1-y^2}$$

with the initial condition

$$y(a) = b.$$

For any given a,
(i)If $|b| > 1$, then $-\sqrt{1-y(a)} = -\sqrt{1-b^2}$ is not a real number. So there is no solution of the differential equation above. (ii)If $|b| < 1$, then $-\sqrt{1-y^2}$ and $\frac{\partial\sqrt{1-y^2}}{\partial y} = \frac{y}{\sqrt{1-y^2}}$ are continuous for (x, y) near the point (a, b). Thus, there exists unique solution of the differential equation with the initial condition $y(a) = b$.
(iii)If $|b| = 1$, there are infinitely many solution of the differential equation. Because the function

$$y = \begin{cases} 1 & \text{if } x \leq c \\ \cos(x-c) & \text{if } c < x < c+\pi \\ -1 & \text{if } x \geq c+\pi \end{cases}$$

are solutions if we choose c appropriately. There are infinitely many possible choices of c for that.
(Example) Consider the problem

$$y' = -\sqrt{1-y^2} \;\; y(0) = 1.$$

Then, for all $c \leq 0$, the function $y = \begin{cases} 1 & \text{if } x \leq c \\ \cos(x-c) & \text{if } c < x < c+\pi \\ -1 & \text{if } x \geq c+\pi \end{cases}$ is a solution of the problem.

1.4.16

The differential equation

$$(x^2+1)(\tan y)\,y' = x$$

is separable, so we proceed accordingly. We have

$$\int \tan y\,dy = \int \frac{x}{x^2+1}\,dx.$$

(Note that there is no risk in dividing by x^2+1, which is never zero.) Integrating, we obtain

$$\log|\sec y| = \frac{1}{2}\log(x^2+1) + C,$$

where C is a constant of integration. (For the integral on the left, write tangent as sine over cosine and make the substitution $v = \cos y$. For the one on the right, make the substitution $u = x^2+1$.) Then

$$|\sec y| = e^{\frac{1}{2}\log(x^2+1)+C} = e^C \cdot e^{\frac{1}{2}\log(x^2+1)} = e^C \cdot e^{\log(x^2+1)^{1/2}} = e^C(x^2+1)^{1/2} = D(x^2+1)^{1/2},$$

where $D = e^C$ is a (positive) constant. The presence of the absolute value around $\sec y$ tells us that a general solution curve is not the graph of a function. We can leave our answer as

$$|\sec y| = D(x^2+1)^{1/2}$$

or, wishing to show some style, we can square both sides of this equation (without introducing extraneous solutions) and write

$$(\sec y)^2 = E(x^2+1),$$

where $E = D^2$ is yet another constant. The answer in the book is not quite correct.

1.4.42

Let $P(t)$ denote the number of potassium atoms in the moon rock t years from the time when it contained only potassium. We know that

$$P(t) = P_0 e^{-kt},$$

where P_0 is the number of potassium atoms at $t = 0$. (Please consult the textbook for the derivation of this equation.) The constant k is related to the half-life τ of potassium by the equation

$$k = \frac{\log 2}{\tau}.$$

(Again, please consult the textbook.) Let $A(t)$ be the number of argon atoms in the rock. Since, on average, one argon atom is created after nine potassium atoms have disintegrated, we have

$$A(t) = \frac{P_0 - P(t)}{9}.$$

(Note that $P_0 - P(t)$ is the number of potassium atoms that have disintegrated. Also, observe that $A(0) = 0$, as expected.) At present, there are equal numbers of potassium and argon atoms in the rock. Suppose that this is T years from the time when the rock contained only potassium. Then

$$P(T) = A(T) = \frac{P_0 - P(T)}{9},$$

so

$$P(T) = \frac{1}{10} P_0.$$

Consequently,

$$P_0 e^{-kT} = \frac{1}{10} P_0.$$

Solving this equation, we obtain

$$T = \frac{\log 10}{k} = \frac{\log 10}{\log 2}\,\tau = \frac{\log 10}{\log 2}\,(1.28 \times 10^9 \text{ yr}) \approx 4.25 \times 10^9 \text{ yr}.$$

The answer in the book is incorrect, again.