MATH320 Homework 2 Solutions

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Section 1.5
26.
We want to find a function $x = x(y)$ that satisfies the equation

$$(1 - 4xy^2)\frac{dy}{dx} = y^3 \tag{1}$$

We note that

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}.$$

So, (1) can be rewritten as

$$\begin{aligned}
&(1 - 4xy^2)\frac{1}{\frac{dx}{dy}} = y^3 \\
\Longrightarrow\ &\frac{dx}{dy} = \frac{1}{y^3} - \frac{4xy^2}{y^3} \\
\Longrightarrow\ &\frac{dx}{dy} + \frac{4}{y}x = \frac{1}{y^3}
\end{aligned}$$

Applying the method of integrating factor,

$$\mu(y) = e^{\int \frac{4}{y}dy} = e^{4\ln y} = y^4 \implies \frac{d}{dy}\Big(xy^4\Big) = y$$

Thus,

$$xy^4 = \int y dy = \frac{1}{2}y^2 + C$$

Hence,

$$\underline{x(y) = \frac{1}{2y^2} + \frac{C}{y^4}}$$

Section 1.5
37. [Notation]

$V(t)$: Volume of water in the tank at time t (sec)
$S(t)$: Salt weight in the tank at time t (sec)

[Note]

$$\Big(\text{Concentration of salt}\Big) = \frac{\Big(\text{Salt weight(lb)}\Big)}{\Big(\text{Volume of water(gal)}\Big)}$$

Or equivalently,

$$\Big(\text{Salt weight(lb)}\Big) = \Big(\text{Concentration of salt(lb/gal)}\Big)\Big(\text{Volume of water(gal)}\Big)$$

$$\begin{cases} \frac{dV}{dt} = \text{gain} - \text{loss} = 5 - 3 = 2; \ \text{ with } \ V(0) = 100 \ gal \\ \frac{dS}{dt} = \text{gain} - \text{loss} = 5 \times 1 - 3 \times \frac{S}{V}; \ \text{ with } \ S(0) = 50 \ lb \end{cases}$$

From the equations above, we have

$$V(t) = 2t + 100.$$

So,

$$\frac{dS}{dt} = 5 - \frac{3}{2t+100}S \ \Leftrightarrow \frac{dS}{dt} + \frac{3}{2t+100}S = 5.$$

By the method of integrating factor,

$$\mu(t) = e^{\int \frac{3}{2t+100}\,dt} = e^{\frac{3}{2}\ln(2t+100)} = (2t+100)^{\frac{3}{2}}$$

and so

$$\frac{d}{dt}\Big[(2t+100)^{3/2}S\Big] = 5(2t+100)^{3/2}$$

By integrating both sides in t,

$$(2t+100)^{3/2} = (2t+100) + \frac{C}{(2t+100)^{3/2}}$$

Using $S(0) = 50$, we have

$$C = -50000.$$

Therefore,

$$S(t) = (2t+100) - \frac{50000}{(2t+100)^{3/2}}$$

The tank becomes full when $V(t) = 2t + 100 = 400$ i.e., $t_{full} = 150$ sec.

Thus,

$$\underline{S(t_{full}) = S(150) = 393.75lb}$$