Review of Basic ODEs

Consider all possible 2nd-order, linear, homogeneous ODEs

$$a(x)\, y''(x) + b(x)\, y'(x) + c(x)\, y(x) = 0$$

in standard form: $y''(x) + p(x)\, y'(x) + q(x)\, y(x) = 0$

with $p(x), q(x)$ real continuous in $x \in (\alpha, \beta)$

Theory The solution space is a 2-dimensional linear function space $\Rightarrow$ we need 2 linearly independent solutions to describe all possible solutions

The initial or boundary conditions select a solution to the physical problem from among the infinite set.

Defn of Linear Independence

$y_1(x)$ and $y_2(x)$ are linearly independent if

$$a_1 y_1(x) + a_2 y_2(x) = 0 \Rightarrow a_1 = a_2 = 0$$

{then $y_2(x)$ is not proportional to $y_1(x)$}

$$\left.\begin{aligned} a_1 y_1(x) + a_2 y_2(x) &= 0 \\ a_1 y_1'(x) + a_2 y_2'(x) &= 0 \end{aligned}\right\} \quad \begin{bmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

has solution $a_1 = a_2 = 0$ if $W(y_1, y_2) \neq 0$ $x \in (\alpha, \beta)$

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2$$

So $y_1(x), y_2(x)$ are linearly independent if $W(y_1, y_2) \neq 0$.

Then $y(x) = c_1 y_1(x) + c_2 y_2(x)$ is the general solution { includes all possible solutions }

Find C_1, C_2 using initial or boundary conditions

$\boxed{\text{Reduction of Order}}$ If you only know one solution $y_1(x)$, a 2^{nd} linearly independent solution can be found as $y_2(x) = y_1(x)\,v(x)$. Find $v(x)$ by plugging in:

$\boxed{\text{Example}}$ $x^2 y'' + 3xy' + y = 0$ Let $y(x) = Ax^r$

$\Rightarrow$ $r(r-1) + 3r + 1 = 0$ $\Rightarrow$ $r = -1$ repeated

$y_1(x) = Ax^{-1}$ $\boxed{\text{check}}$

To find a 2^{nd} linearly independent solution, let

$y(x) = x^{-1} v(x)$ and plug in

$y(x) = x^{-1} v(x)$; $y' = -x^{-2} v + x^{-1} v'$;

$y'' = 2x^{-3} v - 2x^{-2} v' + x^{-1} v''$

Plug into $x^2 y'' + 3xy' + y = 0 \implies$

$$y(x) = x^{-1} \{ C_1 + C_2 \ln x \}$$

$\implies y_2(x) = x^{-1} \ln x$ $\boxed{\text{check}}$

$\boxed{\text{Why does it work?}}$

$$W(y_1, y_1 v) = \begin{vmatrix} y_1 & y_1 v \\ y_1' & v y_1' + y_1 v' \end{vmatrix}$$

$$= y_1 \{ y_1' v + y_1 v' \} - y_1 y_1' v = y_1^2 v'$$

$y_1, y_1 v$ will be linearly independent unless

$y_1^2 v' = 0 \implies y_1^2 = 0$ NO

or $v' = 0 \implies v = C$ NO

Special Equations and Solution Techniques

1. Constant Coefficients $ay'' + by' + cy = 0 \quad -\infty < x < \infty$

$y = Ae^{rx} \implies ar^2 + br + c = 0$

quadratic eqn. for $r \implies$ 2 different real roots,

1 real repeated root (then use RoO),

2 complex conjugate roots

Everything is known!

2. Euler / Equidimensional Eqns.

$ax^2y'' + bxy' + cy = 0 \quad ; \quad y = Ax^r$

$\implies \quad ar(r-1) + br + c = 0$

quadratic eqn. for $r \implies$ Everything is known!

The equation in standard form has a singularity at $x = 0$:

$y'' + \frac{b}{ax}y' + \frac{c}{ax}y = 0$ and $y = Ax^r$ can be singular at $x = 0$.

3. Regular Sturm Liouville Problems

$$-\left[P_{SL}(x)\, y'(x)\right]' + Q_{SL}(x)\, y(x) = \lambda R_{SL}(x)\, y(x) \quad 0<x<1$$

Note: this is not standard form, hence caps and subscript "SL" for coefficients

Note: the eigenvalue λ appears; typically comes from separation of variables as in our heat conduction problem

Separated boundary conditions $a_1 y(0) + a_2 y'(0) = 0$

$$b_1 y(1) + b_2 y'(1) = 0$$

a_1, a_2, b_1, b_2 real

$P_{SL}, P'_{SL}, Q_{SL}, R_{SL}$ real continuous $0 \le x \le 1$

$P_{SL}, R_{SL} > 0$ in $0 \le x \le 1$

Need $P_{SL}(x) > 0$ in $0 \le x \le 1$ for existence [divide by $P_{SL}(x)$ to get standard form]

Need $R_{SL}(x) > 0$ in $0 \le x \le 1$ for reality of eigenvalues λ_n and eigenfunctions $y_n(x)$

SL often written $L[y] = \lambda R_{SL}(x)\, y \quad 0<x<1$

with $L = -\dfrac{d}{dx}\left[P_{SL}(x)\dfrac{d}{dx}\right] + Q_{SL}(x)$

One can prove the following:

** all eigenvalues λ_n are real

** all eigenfunctions $y_n(x)$ are real

** all eigenfunctions satisfy an orthogonality relation with respect to the weight functions $R_{SL}(x)$:

$$\int_0^1 y_n(x)\, y_m(x)\, R_{SL}(x)\, dx = \Omega\, \delta_{nm}$$

Look for proofs in intro ODEs books ...

Recall Bessel of Order Zero:

$r^2 \varphi''(r) + r\varphi'(r) + \lambda r \varphi(r) = 0$, or standard form

$\varphi''(r) + \frac{1}{r}\varphi'(r) + \lambda \varphi(r) = 0$

In SL Form $P_{SL}(r) = r$, $Q_{SL}(r) = 0$, $R_{SL}(r) = r$

$$-\frac{d}{dr}\left[r\frac{d\varphi(r)}{dr}\right] = \lambda r \varphi(r)$$

$$-\{\varphi' + r\varphi''\} = \lambda r \varphi \quad \text{or}$$

$$\varphi'' + \frac{1}{r}\varphi' + \lambda\varphi = 0 \quad \checkmark$$

Bessel is a singular SL problem because

$P_{SL}(0) = R_{SL}(0) = 0$ [not > 0]

but proofs go through with $\varphi(0)$ bounded

Orthogonality in the Familiar setting of sine Functions

Consider $y'' = -\lambda y \quad y(0)=0 \quad y(\pi)=0$

a regular SL problem ...

$y = A\sin(\sqrt{\lambda}\,x) + B\cos(\sqrt{\lambda}\,x)$

$y(0) = 0 \implies B=0$

$y(\pi) = 0 \implies A\sin(\sqrt{\lambda}\,\pi) = 0 \implies \sqrt{\lambda}\,\pi = n\pi$

$\implies \sqrt{\lambda} = n$

$y_n(x) = A_n \sin nx$

Orthogonality : $\displaystyle\int_0^{\pi} \sin nx \sin mx \, dx = \frac{\pi}{2}\delta_{nm}$

To represent a general $f(x)$ odd in $[-\pi, \pi]$ we need all of them!

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$\int_{-\pi}^{\pi} f(x)\sin mx \, dx = \int_{-\pi}^{\pi} \sum_{n=1}^{\infty} b_n \sin nx \sin mx \, dx$$

$$= \pi\delta_{nm} b_n = b_m \implies b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin mx \, dx$$

For eqns. other than constant coefficient or Euler / Equi-dimensional eqns., we typically perform "local analysis" about some point $x = x_0$ [just like we did for Bessel]

We try to approximate the solution near $x = x_0$ with some kind of series solution

$\boxed{\text{3 types of series solutions}}$

(i) $y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$ if x_0 is an "ordinary point."

where n is integer and we need to find the a_n's.

This Taylor series solution is not singular at $x = x_0$

(ii) $y(x) = (x - x_0)^{\alpha} \sum_{n=0}^{\infty} a_n (x - x_0)^n$ if x_0 is a "regular singular point."

n integer, α is real ; we need to find the a_n's and α.

This is called a Frobenius series

The solution may be singular at $x = x_0$ depending on the value of α, but the singularity is a power of $(x-x_0)$

(iii) $y(x) = \exp\left[\beta(x-x_0)\right] \sum_{n=0}^{\infty} a_n (x-x_0)^{\gamma n}$

If x_0 is an "irregular singular point"

n integer, $\gamma > 0$ real, $\beta = \beta(x)$;

we need to find $\beta(x)$, γ and the a_n's.

Now the solution may be singular at $x = x_0$ and the singularity may cause $y(x) \to \infty$ faster than a power of $(x-x_0)$

Note the difference between

$y(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^{\alpha+n}$ Frobenius ; integer powers apart from an overall factor of $(x-x_0)^{\alpha}$

$y(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^{\gamma n}$ which may be non-integer powers

In all cases we need to ask: For what values of x does the expansion give a good approximation to the solution?

How many terms do we need to keep in order to obtain a good approximation?

In partular, does the series converge?

Pointwise Convergence $\Rightarrow$

$$\lim_{N \to \infty} \left| y(x) - \sum_{n=0}^{N} a_n (x - x_0)^n \right| = 0$$

For fixed x as $N \to \infty$

If the series does not converge, can it still be useful?