$$X' = \begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix} X + \begin{pmatrix} 4\ln t \\ t^{-1} \end{pmatrix}$$

Homogeneous.

Eigenvalue: $(2-\lambda)(-2-\lambda)+4=0 \Rightarrow \lambda=0$ repeated.

Eigenvector: For $\lambda=0$. $\left[\begin{array}{cc|c} 2 & -4 & 0 \\ 1 & -2 & 0 \end{array}\right] \Rightarrow \mathbf{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

Soln: there's only one eigenvector. thus soln is $e^{0t}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$, which is $X_1=\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

Another soln should be

$$X_2 = te^{0t}\,\xi + e^{0t}\,\psi$$

w/ $\begin{cases} \xi \text{ is eigenvector} \\ \psi: \quad A\psi = \xi \end{cases} \Rightarrow \begin{cases} \xi = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\ \left[\begin{array}{cc|c} 2 & -4 & 2 \\ 1 & -2 & 1 \end{array}\right] \Rightarrow \psi = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{cases}$

thus $X_2 = \begin{pmatrix} 2t+1 \\ t \end{pmatrix}$

thus soln to homogeneous part is $C_1\begin{pmatrix} 2 \\ 1 \end{pmatrix} + C_2\begin{pmatrix} 2t+1 \\ t \end{pmatrix}$.

Inhomo: let $X_p = u_1 X_1 + u_2 X_2$.

hope $X_p' = AX_p + f$

$$(u_1X_1 + u_2X_2)' = Au_1X_1 + Au_2X_2 + f$$

$$u_1'X_1 + u_2'X_2 + \cancel{u_1X_1'} + \cancel{u_2X_2'} = \cancel{u_1X_1'} + u_2\cancel{X_2'} + f$$

$\therefore \quad [X_1 \quad X_2]\cdot\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = [f]$

$$\left[\begin{array}{cc|c} 2 & 2t+1 & 4\ln t \\ 1 & t & t^{-1} \end{array}\right] \xrightarrow{R_1 - 2R_2} \left[\begin{array}{cc|c} 0 & 1 & 4\ln t - \frac{2}{t} \\ 1 & t & \frac{1}{t} \end{array}\right]$$

$$\Rightarrow \left.\begin{array}{l} u_2' = 4\ln t - \frac{2}{t} \\ u_1' = -4t\ln t + 2 + \frac{1}{t} \end{array}\right\} \Rightarrow \begin{array}{l} u_2 = 4t\ln t - 4t - 2\ln t \\ u_1 = -2t^2\ln t + t^2 + 2t + \ln t. \end{array}$$

$$\therefore \quad X_p = u_1X_1 + u_2X_2$$
$$= \begin{pmatrix} 4t^2\ln t - 6t^2 \\ 2t^2\ln t - 3t^2 + 2t + \ln t - 2t\ln t \end{pmatrix}$$

General: $X_g = C_1\begin{pmatrix} 2 \\ 1 \end{pmatrix} + C_2\begin{pmatrix} 2t+1 \\ t \end{pmatrix} + X_p$