Math 320 (Smith): Exam 1

Feb. 24, 2009

YOUR NAME:

It is a very good idea to write your name on every page.

YOUR TA's NAME AND SECTION NUMBER:

Prob 1 /20	
Prob 2 /20	
Prob 3 /20	
Prob 4 /20	
Prob 5 /20	
TOTAL /100	

1. (20 points) Consider

$$a(x)y' - m(x)y = s(x), \quad y(x_o) = y_o, \quad x_o < 0, \quad y_o > 0, \tag{1}$$

where $a(x), m(x), s(x)$ are real and continuous on $-\infty < x < \infty$,
with $a(x)$ nonzero for $x \neq 0$ and $a(0) = 0$.

(a) Find the solution in terms of $a(x), m(x), s(x), x_o$ and y_o.

(b) Give the range of validity of the solution.

2. (20 points) Consider

$$\frac{dy}{dt} = (y-p)(y-k)^2, \quad y(0) = y_o, \quad t \geq 0, \quad -\infty < y < \infty, \quad k > p > 0. \tag{2}$$

(a) Sketch, roughly, a direction field and classify all critical points.

(b) Determine (from your sketch), the asymptotic behavior of the solution as $t \to \infty$ for $p < y_o < k$.

3. (20 points) Write the following system as $\mathbf{Ax} = \mathbf{b}$ and determine for what values of k the system has (i) a unique solution, (ii) no solution, and (iii) infinitely many solutions. In the case of (i) and/or (iii), find the solution(s).

$$
\begin{aligned}
x_1 + 3x_3 &= 1 \\
-x_1 + 2x_2 - x_3 &= k \\
3x_1 + x_2 + 10x_3 &= 4
\end{aligned}
\tag{3}
$$

4. (20 points) Solve

$$(x^2 - 4)y' = 2y, \quad y(-1) = 1 \tag{4a}$$

Give the range of validity of the solution.

5. (20 points) Given

$$\frac{dy}{dx} = y^2, \quad y(0) = 1. \tag{5}$$

(a) Find the exact solution and state the region of validity of the exact solution.

(b) Find the Taylor series expansion for $y(x_o + h)$ (write down the first four terms).

(c) Use one step of the Improved Euler method with step size h to find an approximation for $y(x_o + h)$. Express your approximation $\tilde{y}(x_o + h)$ as a function of h only.

(d) How many terms in the Taylor series expansion for the exact solution $y(x_o + h)$ are the same as the Improved Euler approximate solution $\tilde{y}(x_o + h)$? **Explain your answer**. (In case you cannot complete part (a), you should not need to calculate explicitly the Taylor series expansion for the exact solution).

(e) In general for a well-posed initial value problem $y' = f(x, y)$, $y(x_o) = y_o$: if the step size is decreased from h to $h/10$, how will the **global** error change for the Improved Euler method?