Exam 1 320

1. $(x^2-1)y' - xy = x(1-x^2) \qquad y(0)=2$

$$y' - \frac{x}{x^2-1}\,y = -x$$

$$\mu(x) = \exp\left[-\int^x \frac{t}{t^2-1}\,dt\right] = \exp\left[-\tfrac{1}{2}\ln|x^2-1|\right]$$

$$= (1-x^2)^{-\frac{1}{2}}$$

$$\frac{d}{dx}\left[(1-x^2)^{-\frac{1}{2}}y\right] = \frac{-x}{(1-x^2)^{\frac{1}{2}}}$$

$$(1-x^2)^{-\frac{1}{2}}y = (1-x^2)^{\frac{1}{2}} + C$$

$$y = 1-x^2 + C(1-x^2)^{\frac{1}{2}}$$

$$y(0) = 2 \implies 2 = 1+C \implies C = \underline{1}$$

$$\boxed{\begin{array}{c} y = 1-x^2 + (1-x^2)^{\frac{1}{2}} \\ -1 \le x \le 1 \end{array}}$$

2. $\frac{dy}{dt} = y(y^2-p^2) = y(y+p)(y-p) \qquad p>1$

$= (y+p)\,y\,(y-p) \qquad y(0)=y_0 \quad t\geq 0$

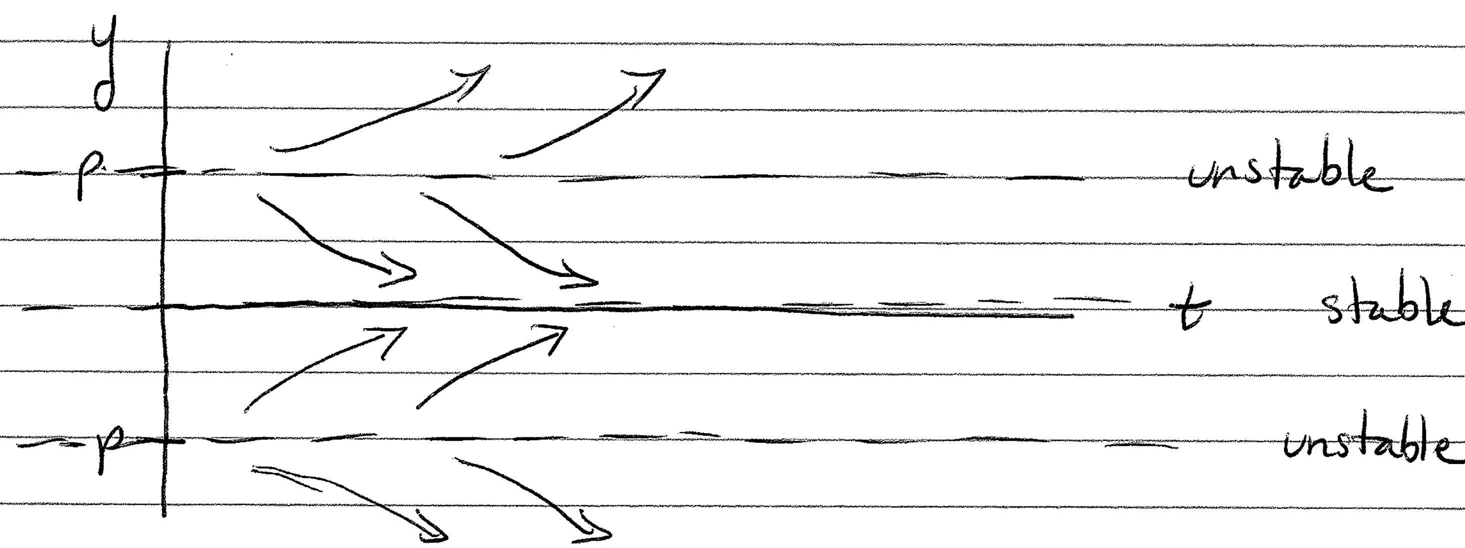
$y<-p \qquad \frac{dy}{dt} = - - - \quad = -$

$-p<y<0 \qquad \frac{dy}{dt} = + - - \quad = +$

$0<y<p \qquad \frac{dy}{dt} = + + - \quad = -$

$y>p \qquad \frac{dy}{dt} = + + + \quad = +$

critical points $y=-p,\ y=0,\ y=p$



For $y_0>p \qquad \lim_{t\to\infty} y(t) \to \infty$

3. Last row $0x_1 + 0x_2 + (k+2)x_3 = 0$

$\Rightarrow (k+2)x_3 = 0$

(i) unique soln. for $k \neq -2$. Then

$x_3 = 0, \quad x_2 = 2, \quad x_1 = 4 - 2x_2 = 0$

$$\underline{x} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

(ii) No value of $k \Rightarrow$ no solution

(iii) $k = -2 \Rightarrow$ infinitely many solutions

Let $x_3 = t$

From row 2: $x_2 = 2 - 2t$

From row 1: $x_1 = 4 - 3t - 2(2 - 2t) = t$

$$\underline{x} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

4 $\dfrac{dy}{dx} = xy^3 \quad y(0) = -1$

a $\dfrac{dy}{y^3} = x\,dx, \quad -\frac{1}{2}y^{-2} = \frac{1}{2}x^2 + C$

$y^{-2} = -x^2 + C^*, \quad 1 = 0 + C^*$

$\dfrac{1}{y^2} = 1 - x^2 \implies y^2 = \dfrac{1}{1-x^2}$

$y = \pm\left(\dfrac{1}{1-x^2}\right)^{1/2}$ choose $-$ sign based on $y(0) = -1$

$$\boxed{y = -\left(\frac{1}{1-x^2}\right)^{1/2}} \qquad \boxed{-1 < x < 1}$$

b $\tilde{y}_1 = y_0 + \frac{h}{2}\left\{x_0 y_0^3 + x_1\left[y_0 + h x_0 y_0^3\right]^3\right\}$

$\tilde{y}_1(h) = -1 + \frac{h}{2}\left\{0 + h(-1) + 0\right\} = -1 - h^2/2$

c First 2 terms in Taylor series for exact soln y about $x_0 = 0$ (see next page)

$$y(h) = -\left(\frac{1}{1-h^2}\right)^{\frac{1}{2}} = -1 - \frac{h^2}{2} + \cdots$$

Reasoning: The constant, h, h^2 terms in the expansion for $y(h)$ must match the Modified Euler expression, since Modified Euler has local truncation error $O(h^3)$

Calculation: (not necessary!)

$$\frac{1}{1-h^2} = 1 + h^2 + h^4 + \cdots \qquad |h| < 1$$

$$\left(\frac{1}{1-h^2}\right)^{\frac{1}{2}} = 1 + \frac{h^2}{2} + \cdots \qquad |h| < 1$$

$$-\frac{1}{(1-h^2)^{1/2}} = -1 - \frac{h^2}{2} + \cdots$$