Exam Solutions

(1) $\dfrac{dy}{dt} = (y-a)(y+b)(y-b) \qquad y(0) = -a$

$$b > a > 0$$

$$\frac{dy}{dt} = (y+b)(y-a)(y-b)$$

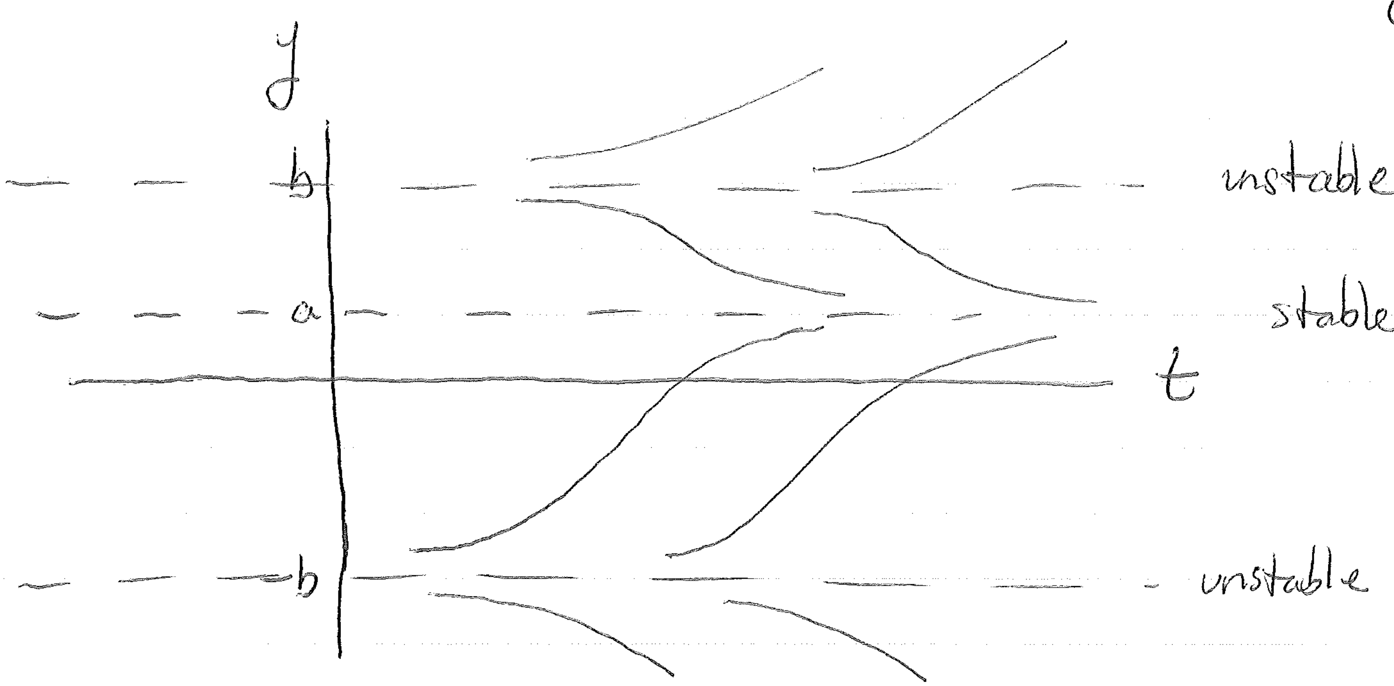
critical points $\quad y=-b, \; y=a, \; y=b$

$y < -b \qquad \dfrac{dy}{dt} = - - - \; < 0$

$-b < y < a \qquad \dfrac{dy}{dt} = + - - \; > 0$

$a < y < b \qquad \dfrac{dy}{dt} = + + - \; < 0$

$y > b \qquad \dfrac{dy}{dt} = + + + \; > 0$



with initial condition $y(0) = -a$

$$\lim_{t \to \infty} y(t) = a$$

(2) $\begin{bmatrix} -1 & 1 & 1 & 2 \\ 0 & 5 & -k & 4 \\ 0 & 0 & k & p+3 \end{bmatrix}$

(a) $p = -3$ $\qquad kx_3 = 0$

$k = 0 \implies$ infinite number of solutions

$k \neq 0 \implies$ unique solution

never no solution

$k = 0$ $\qquad x_3 = 0$

$5x_2 - kx_3 = 4 \quad , \quad x_2 = 4/5$

$-x_1 + x_2 + x_3 = 2$

$-x_1 + \frac{4}{5} = \frac{10}{5} \quad , \quad x_1 = \frac{6}{5}$

$\underline{x} = \begin{bmatrix} 6/5 \\ 4/5 \\ 0 \end{bmatrix}$

$k = 0 \qquad x_3 = t$

$5x_2 = 4 \quad , \quad x_2 = 4/5$

$-x_1 + x_2 + x_3 = 2$

$-x_1 = 2 - \frac{4}{5} - t = \frac{6}{5} - t$

$x_1 = t - 6/5$

$$\underline{x} = \begin{bmatrix} t - 6/5 \\ 4/5 \\ t \end{bmatrix} = \begin{bmatrix} -\frac{6}{5} \\ \frac{4}{5} \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

(b) $p = -2 \qquad kx_3 = 1$

$k = 0$ no solution

$k \neq 0$ unique solution

never an infinite number of solutions

(3) $\frac{dy}{dx} = \frac{-y}{(x-1)} + \frac{e^{-x}}{(x-1)} \qquad y(0) = 2$

standard form:

$$\frac{dy}{dx} + \frac{y}{(x-1)} = \frac{e^{-x}}{x-1}$$

integrating factor $u(x) = \exp\left[\int \frac{1}{(x-1)}\, dx\right]$

$$= \exp\left[\ln|x-1|\right] = (1-x)$$

$$\frac{d}{dx}\left[(1-x)y\right] = -e^{-x}$$

$$(1-x)y = e^{-x} + C$$

$$y = \frac{e^{-x}}{(1-x)} + \frac{C}{(1-x)}, \quad 2 = 1 + C \Rightarrow C = \underline{1}$$

$$\boxed{y = \frac{e^{-x}}{(1-x)} + \frac{1}{(1-x)}, \quad x < 1}$$

(b) $\tilde{y}_1 = y_0 + hF(x_0, y_0)$

$$= y_0 + h\left\{\frac{-y_0}{x_0-1} + \frac{e^{-x_0}}{x_0-1}\right\}$$

$$= 2 + h\left\{\frac{-2}{-1} + \frac{1}{-1}\right\}$$

$$= 2 + h\{2-1\} = 2+h$$

$$\boxed{\tilde{y}(h) = 2+h}$$

Exam Solutions

4 (a) $\frac{dy}{dx} = -\frac{5}{2} x^4 y^3, \quad y(0) = -1$

$$\frac{dy}{y^3} = -\frac{5}{2} x^4 \, dx$$

$$-\frac{1}{2} y^{-2} = -\frac{5}{2} \frac{x^5}{5} + C$$

$$y^{-2} = x^5 + C^* \quad , \quad C^* = 1$$

$$y^2 = \frac{1}{x^5+1}$$

$$\boxed{y = -\frac{1}{(1+x^5)^{1/2}} \qquad x > -1}$$

(b) $\tilde{y}_1 = y_0 + \frac{h}{2}\left\{ f(x_0, y_0) + f(x_1, \tilde{S}_0) \right\}$

$$\tilde{S}_0 = y_0 + hf(x_0, y_0) = -1 + h\left(-\frac{5}{2}\right) x_0^4 y_0^3$$

$$= -1$$

$$\tilde{y}_1 = -1 + \frac{h}{2}\left\{ \left(-\frac{5}{2}\right) x_0^4 y_0^3 + \left(-\frac{5}{2}\right) x_1^4 \tilde{S}_0^3 \right\}$$

$$= -1 + \frac{h}{2}\left\{ 0 - \frac{5}{2} h^4 (-1)^3 \right\}$$

$$\boxed{\tilde{y}(h) = -1 + \frac{5}{4} h^5}$$

(5) TRUE