Exam 2

1. $2y'' + \frac{8}{x}y' + \frac{9}{2x^2}y = 0 \qquad x > 0$

(a) $4x^2y'' + 16x\,y' + 9y = 0 \qquad \text{Try } y = Ax^r$

$\Longrightarrow \quad 4r(r-1) + 16r + 9r = 4r^2 + 12r + 9 = 0$

$4\left(r^2 + 3r + \frac{9}{4}\right) = 0$

$$r = \frac{-3 \pm \sqrt{9 - 4\cdot 9/4}}{2} = \frac{-3}{2} \quad \text{repeated}$$

$$y = C_1 x^{-\frac{3}{2}} + C_2(\ln x)\,x^{-3/2}$$

(b) $y' = -\frac{3}{2}C_1 x^{-5/2} + C_2\left[\frac{x^{-3/2}}{x} - \frac{3}{2}\ln x \; x^{-5/2}\right]$

$$\begin{bmatrix} 1 & 0 \\ -\frac{3}{2} & 1 \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

2) $\boxed{c_1 \underline{u} + c_2 \underline{v} = 0 \quad \Rightarrow \quad c_1 = c_2 = 0}$

since $\underline{u}, \underline{v}$ linearly independent (given)

Check linear independence / dependence of
$\underline{u} - \underline{v}$, $\underline{v}$

* $A_1(\underline{u} - \underline{v}) + A_2 \underline{v} = 0$

$A_1 \underline{u} - A_1 \underline{v} + A_2 \underline{v} = 0$

$A_1 \underline{u} + (A_2 - A_1)\underline{v} = 0$

* $\Rightarrow \quad A_1 = 0 \qquad A_2 - A_1 = 0$

since $\underline{u}, \underline{v}$ linearly independent

$\boxed{\Rightarrow \quad A_1 = A_2 = 0 \quad \Rightarrow \quad \underline{u} - \underline{v}, \underline{v} \text{ linearly independent}}$

3 $\left[\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{1}{2} & b_1 \\ -2 & a_{22} & -1 & b_2 \\ 4 & 2 & 2 & b_3 \end{array}\right] \begin{array}{l} \\ R_2+2R_1 \\ R_3-4R_1 \end{array}$

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} & b_1 \\ 0 & a_{22}+1 & 0 & b_2+2b_1 \\ 0 & 0 & 0 & b_3-4b_1 \end{bmatrix}$$

(i) $a_{22} \neq -1$, 2-parameter subspace

$b_3 = 4b_1 = 4s \implies \underline{b} = s\begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

(ii) $a_{22} = -1$, 1-parameter subspace

$b_3 = 4b_1$, $b_2 = -2b_1$ $\quad \underline{b} = s\begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$

(iii) Never

(b) (i) $\begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$

(c) Yes $\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$x_3 = p$, $x_2 = 2$, $x_1 + \frac{1}{2}x_2 + \frac{1}{2}x_3 = 1$

$x_1 = 1 - 1 - \frac{p}{2} = -\frac{p}{2}$

$$\underline{x} = p\begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

4) $xy'' - (1+x)y' + y = 0 \quad , \quad x > 0$

$y_1 = e^x \quad$ Let $\quad y = ue^x \;,\; y' = u'e^x + ue^x$

$y'' = u''e^x + 2u'e^x + ue^x$

$x(u'' + 2u' + u) - (1+x)(u' + u) + u = 0$

$xu - (1+x)u + u = 0 \quad \Rightarrow$

$xu'' + 2xu' - (1+x)u' = 0$

$xu'' + (x-1)u' = 0 \quad , \quad u'' + \left(1 - \frac{1}{x}\right)u' = 0$

Let $\quad w = u' \quad \Rightarrow \quad \frac{dw}{w} = \left(\frac{1}{x} - 1\right)dx$

$\ln|w| = (\ln x - x + C) \quad , \quad w = c^* x e^{-x}$

$u = c^*(-xe^{-x} - e^{-x}) + D \quad$ (integration by parts)

Check: $\frac{d}{dx}(-xe^{-x} - e^{-x}) = -e^{-x} + xe^{-x} + e^{-x} = xe^{-x}$

$y = c^*(-x-1) + De^x = c'(x+1) + De^x$

5 $y^{(vi)} - 3y^{(5)} + 3y^{(4)} - 9y''' + 24y'' - 24y' + 8y = 0$

$(r-1)^3(r^3 - 8) = 0$ $\quad r = 1$ repeated

$r = 2(1)^{1/3}$

$m=0 \quad (1)^{1/3} = e^{2\pi i/3} = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$

$m=1 \quad (1)^{1/3} = e^{4\pi i/3} = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$

$m=2 \quad (1)^{1/3} = e^{6\pi i/3} = \underline{1}$

$$y = c_1 e^{x} + c_2 x e^{x} + c_3 x^2 e^{x} + c_4 e^{2x} + c_5 e^{-x}\cos\sqrt{3}x + c_6 e^{-x}\sin\sqrt{3}x$$