

5.4.

2L.

$$m x'' + c x' + k x = 0$$

$$m: 12 \text{ lbs} \rightarrow 12 \cdot \frac{1}{32} \text{ slugs} \quad (32 \text{ lbs} \rightarrow 1 \text{ slug}).$$

$$m = \frac{12}{32} = \frac{3}{8} \text{ slugs} (= \text{lbs} \cdot \text{sec}^2 / \text{ft})$$

$$c = 3 \text{ (lbs} \cdot \text{sec / ft)}$$

$$k: \text{ Hook's law} \quad 12 \text{ lbs} = k \cdot (0.5 \text{ ft}) \\ \Rightarrow k = 24 \text{ (lbs/ft)}$$

$$\text{So, } \frac{3}{8} x'' + 3 x' + 24 x = 0$$

$$\Rightarrow x'' + 8 x' + 64 x = 0$$

$$(a) \quad x(0) = 1 \text{ (ft.)} \quad x'(0) = 0 \text{ (ft./sec)}$$

$$r^2 + 8r + 64 = 0$$

$$r = -4 \pm 4\sqrt{3}i$$

$$x(t) = e^{-4t} (C_1 \cos(4\sqrt{3}t) + C_2 \sin(4\sqrt{3}t))$$

$$x'(t) = -4e^{-4t} (C_1 \cos(4\sqrt{3}t) + C_2 \sin(4\sqrt{3}t)) + 4\sqrt{3}e^{-4t} (-C_1 \sin(4\sqrt{3}t) + C_2 \cos(4\sqrt{3}t))$$

$$x(0) = C_1 = 1$$

$$x'(0) = -4C_1 + 4\sqrt{3}C_2 = 0$$

$$C_2 = \frac{1}{\sqrt{3}}$$

Thus, $x(t) = e^{-4t} \left(\cos(4\sqrt{3}t) + \frac{1}{\sqrt{3}} \sin(4\sqrt{3}t) \right)$

(b) Write $x(t) = Ae^{-4t} \cos(4\sqrt{3}t - \omega)$
and find A and ω .

$$A = \sqrt{1^2 + \left(\frac{1}{\sqrt{3}}\right)^2} = \frac{2}{\sqrt{3}}$$

$$\cos\omega = \frac{1}{\left(\frac{2}{\sqrt{3}}\right)}, \quad \sin\omega = \frac{\frac{1}{\sqrt{3}}}{\frac{2}{\sqrt{3}}} \Rightarrow \tan\omega = \frac{1}{\sqrt{3}}$$

$$\omega = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$$

Therefore, $x(t) = \frac{2}{\sqrt{3}} e^{-4t} \cos\left(4\sqrt{3}t - \frac{\pi}{6}\right)$.

Frequency: $4\sqrt{3}$ rad/sec.

Amplitude: $\frac{2}{\sqrt{3}} e^{-4t}$

Phase angle: $\frac{\pi}{6}$.

* If $c=0$, $\frac{3}{8}x'' + 24x = 0$
(Undamped motion) $\Rightarrow x'' + 64x = 0$
 $r^2 + 64 = 0 \Rightarrow r = \pm 8i$
 $x(t) = c_1 \cos 8t + c_2 \sin 8t$
 $x(0) = c_1 = 1$

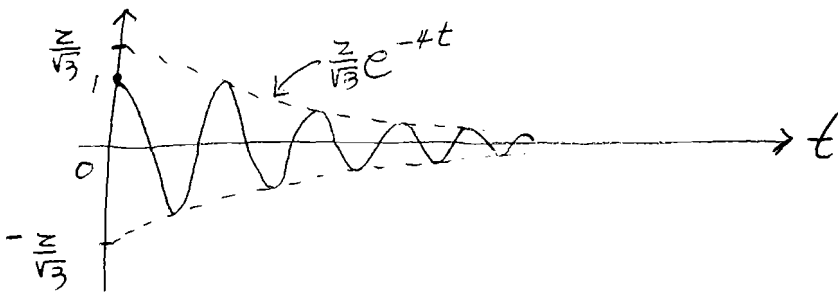
$$x'(0) = \delta C_2 = 0 \Rightarrow C_2 = 0.$$

Thus, $x(t) = \cos \delta t.$

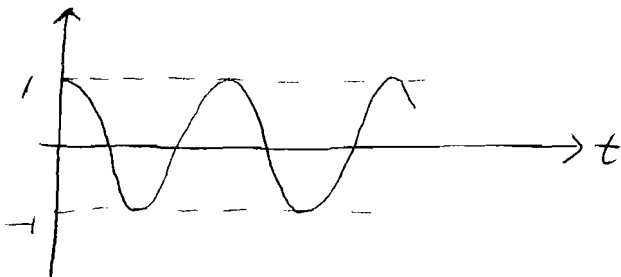
Frequency:	δ rad/sec
Amplitude:	1
Phase angle:	0.

Graph:

$$x(t) = \frac{2}{\sqrt{3}} e^{-4t} \cos\left(4\sqrt{3}t - \frac{\pi}{6}\right)$$



$$x(t) = \cos \delta t$$



5.5 18. Find y_p .

$$y^{(4)} - 5y'' + 4y = e^x - xe^{2x}$$

Step 1. Find y_h .

$$\text{Characteristic equation: } r^4 - 5r^2 + 4 = (r^2 - 4)(r^2 - 1) = 0$$

$$r = \pm 1, \pm 2$$

$$\text{Thus, } y_h = c_1e^{-x} + c_2e^x + c_3e^{-2x} + c_4e^{2x}$$

Step 2. Set $y_p = Axe^x + x(Bx + D)e^{2x}$

because e^x and e^{2x} are in y_h already.

Let us find A, B, D .

$$y_p' = Ae^x(x+1) + e^{2x}\{2Bx^2 + 2(B+D)x + D\}$$

$$y_p'' = Ae^x(x+2) + e^{2x}\{4Bx^2 + 4(2B+D)x + 2(2D+B)\}$$

$$y_p^{(3)} = Ae^x(x+3) + e^{2x}\{8Bx^2 + x(3B+D)x + 12(B+D)\}$$

$$y_p^{(4)} = Ae^x(x+4) + e^{2x}\{16Bx^2 + 16(4B+D)x + 16(3B+2D)\}$$

Then,

$$\begin{aligned} & y_p^{(4)} - 5y_p'' + 4y_p \\ &= -6Ae^x + e^{2x}\{24Bx + (38B + 12D)\} \\ &\stackrel{\text{want}}{=} e^x - xe^{2x} \end{aligned}$$

By comparing each coefficients, we obtain

$$-6A = 1$$

$$24B = -1$$

$$38B + 12D = 0$$

So,

$$A = -\frac{1}{6}, B = -\frac{1}{24}, D = \frac{19}{244}$$

Hence, we finally obtain

$$y_p = -\frac{1}{6}xe^x + xe^{2x}\left(-\frac{1}{24}x + \frac{19}{244}\right).$$

Note. General solution $y = y_h + y_p$.

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$$y'' - 6y' + 13y = xe^{3x} \sin 2x$$

Step 1. y_h :

$$r^2 - 6r + 13 = 0 \Rightarrow r = 3 \pm 2i$$
$$\text{So, } y_h = e^{3x}(c_1 \cos 2x + c_2 \sin 2x)$$

Step 2. y_p

$$y_p = x(\text{Polynomial of degree 1}) \cdot e^{3x}(A \sin 2x + B \cos 2x)$$
$$= x(Dx + E)e^{3x}(A \sin 2x + B \cos 2x)$$

because $e^{3x} \cos 2x$ and $e^{3x} \sin 2x$ are in y_h already. We can simplify the expression above as

$$y_p = (ax^2 + bx)e^{3x} \sin 2x + (gx^2 + hx)e^{3x} \cos 2x$$

for simplicity.

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$$y'' + y' + y = \sin x \sin 3x$$

First of all, we note that

$$\cos(a + b) = \cos a \cos b - \sin a \sin b$$
$$\cos(a - b) = \cos a \cos b + \sin a \sin b$$

So,

$$\cos(a + b) - \cos(a - b) = -2 \sin a \sin b$$
$$\Rightarrow \sin a \sin b = -\frac{1}{2}(\cos(a + b) - \cos(a - b))$$

So, we can write

$$\sin 3x \sin x = -\frac{1}{2}(\cos 4x - \cos 2x)$$

Step 1.

$$y_h = e^{-\frac{x}{2}}(c_1 \cos \frac{\sqrt{3}}{2}x + c_2 \sin \frac{\sqrt{3}}{2}x)$$

Step 2. So we can set

$$y_p = (A \cos 4x + B \sin 4x) + (D \cos 2x + E \sin 2x)$$

Then,

$$y_p'' + y_p' + y_p = (-16A + 4B + A) \cos 4x + (-16B - 4A + B) \sin 4x + (-4D + 2E + D) \cos 2x + (-4E - 2D + E) \sin 2x$$
$$\stackrel{\text{want}}{=} -\frac{1}{2} \cos 4x + \frac{1}{2} \cos 2x$$

So,

$$\begin{cases} -15A + 4B = -\frac{1}{2} \\ -4A - 15B = 0 \end{cases} \quad \begin{cases} -3D + 2E = \frac{1}{2} \\ -2D - 3E = 0 \end{cases}$$

From these equations, we have

$$A = -\frac{15}{422}, B = \frac{2}{211} \quad D = -\frac{3}{26}, E = \frac{1}{13}$$

Thus,

$$y_p = -\frac{15}{422} \cos 4x + \frac{2}{211} \sin 4x - \frac{3}{26} \cos 2x + \frac{1}{13} \sin 2x$$

General solution:

$$y = y_h + y_p = e^{-\frac{x}{2}} \left(c_1 \cos \frac{\sqrt{3}}{2} x + c_2 \sin \frac{\sqrt{3}}{2} x \right) - \frac{15}{422} \cos 4x + \frac{2}{211} \sin 4x - \frac{3}{26} \cos 2x + \frac{1}{13} \sin 2x$$

If you find any typo or mistake, please let me(Jean) know by email:[bae\(at\)math\(dot\)wisc\(dot\)edu](mailto:bae@math.wisc.edu)