On near-resonances and symmetry breaking in rotating turbulence at moderate Rossby numbers

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Motivation

A predominance of cyclones in the atmosphere and oceans

Figure 1: Hurrican Ivan, Sept. 04
Pedlosky (1986, Chapter 1) estimates:

- \( Ro \equiv \frac{U}{Lf} \approx 0.14 \) for typical synoptic-scale winds at mid-latitudes, with
  \[
  U \approx 20 \text{ m s}^{-1}, \quad L \approx 1,000 \text{ km}, \quad f \equiv 2\Omega, \quad \Omega \approx 7.3 \times 10^{-5} \text{s}^{-1}
  \]
- \( Ro \approx 0.07 \) in the Gulf Stream
  \[
  U \approx 100 \text{ cm s}^{-1}, \quad L \approx 100 \text{ km}
  \]
Cyclonic vortices in physical and numerical experiments

Equations and properties of homogeneous, rotating flows

Reduced Models: motivation and definitions

Numerical simulations: reduced models of 3D rotation

Analogous treatment of modified $\beta$-plane flow

Conclusions and discussion
Motivation

- **Numerical studies with forcing at intermediate scales**

- **Numerical decay**
  Bartello, Metais & Lesieur (1994)

- **Rotating tank/annulus experiments**
  Hopfinger, Browand & Gagne (1982), Longhetto et al. (2002), Baroud et al. (2003)
Cyclones in inhomogeneous flows with forcing

- Numerical simulations with confinement in the \( \hat{z} \)-direction and forcing localized in physical space (Godeferd & Lollini 1999)

Cyclones dominate for \( Ro_l \equiv U/(Lf) \approx [0.2 - 0.3] \) based on a local \( U \) and \( L \) away from the forcing
Numerical simulations with confinement in the $\hat{z}$-direction and forcing localized in physical space (Godeferd & Lollini 1999)

Cyclones dominate for $Ro_l \equiv \frac{U}{Lf} \approx [0.2 - 0.3]$ based on a local $U$ and $L$ away from the forcing.

Tank experiments with a grid oscillating at frequency $n$ (Hopfinger, Browand & Gagne 1982)

Cyclones dominate for smaller grid Rossby numbers $Ro_g \approx 3$ and local $Ro_l \approx 0.05$, with $Ro_g \equiv \frac{n}{f} \approx 3 - 33$. 
Baroud, Plapp, Swinney & She (2003)

A rotating annulus forced by pumping water into (out of) an inner (outer) ring of holes on the bottom creates a nearly 2D flow in which vortices are advected by a counter-rotating azimuthal jet

$$Ro = \frac{\omega_{rms}}{f} \approx 0.1$$

Cyclones more likely to preserve coherence
Cyclones in homogeneous decay

• LES by Bartello, Metais & Lesieur 1994 show a range of initial $Ro_i \approx [0.1, 0.4]$ with efficient transfer of energy from 3D to 2D modes maximal asymmetry between cyclones/anticyclones $Ro_i$ based on the rms vorticity in the $\hat{z}$-direction
Cyclones in homogeneous decay

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- Lab experiments by Longehtto et al. 2002 in the Grenoble “Coriolis” tank with $Ro_i \approx 1$ cyclones stronger and longer-lived than anticyclones $Ro_i$ based on speed and teeth spacing of the rake used to create the initial turbulence
Navier–Stokes in a frame rotating at rate $\Omega = \Omega \hat{z}$

$$\frac{\partial u}{\partial t} + \frac{1}{Ro} \hat{z} \times u + (\nabla \times u) \times u = -\nabla p + \frac{1}{Re} \nabla^2 u + f$$

$$\nabla \cdot u = 0$$

$$Ro = \frac{U}{fL}, \quad f = 2\Omega, \quad Re = \frac{UL}{\nu}$$
Inertial waves

The unforced, linear, inviscid limit $\implies$ inertial waves

$$u(x, t; k, s) = h_s(k) \exp\left[i \left( k \cdot x - \sigma_s(k) \frac{t}{R_0} \right)\right] + \text{c.c.}$$

frequencies: $\sigma_s(k) = s \frac{k_z}{k}, \quad s = \pm 1$

eigenmodes: $h_s(k) = \hat{k} \times \hat{\phi} + is\hat{\phi}, \quad \phi = \frac{k \times \hat{z}}{|k \times \hat{z}|}$

$$h_s((0, 0, k_z)) = \frac{1}{2} (1 + is \text{sgn}(k_z), 1 - is \text{sgn}(k_z), 0)$$
The linear eigenmodes are *helical* since

\[ \nabla \times \mathbf{u} \propto \mathbf{u} \]

In $k$-space:

\[ ik \times \mathbf{h}_s = -sk \mathbf{h}_s \]
Since $h_s(k)$ form an orthogonal basis

$$u(x, t) = \sum_k \sum_s b_s(t, k) h_s(k) \exp \left[ i \left( k \cdot x - \sigma_s(k) \frac{t}{Ro} \right) \right]$$

and Navier–Stokes becomes

$$\left( \frac{\partial}{\partial t} + \nu k^2 \right) b_{sk}(t; k)$$

$$= \sum_{\Delta_{k,p,q}} \sum_{s_p, s_q} C_{k,p,q}^{s_k, s_p, s_q} b_{s_p}^*(t; p) b_{s_q}^*(t; q) \exp \left[ i \left( \sigma_{sk} + \sigma_{sp} + \sigma_{sq} \right) \frac{t}{Ro} \right]$$

$$C_{k,p,q}^{s_k, s_p, s_q} = (s_p p - s_q q) (h_{s_p}^* \times h_{s_q}^*) \cdot h_{s_k}^*$$
Exactly and nearly resonant triad interactions

For $Ro \to 0$, exactly resonant triads dominate, with

$$\sigma_{sk} + \sigma_{sp} + \sigma_{sq} = 0$$

For moderate (small) $Ro$ on time scales $T = O(1/Ro)$, near-resonant triads are important (Newell 1969), with

$$|\sigma_{sk} + \sigma_{sp} + \sigma_{sq}| = O(Ro), \quad \sigma_{sk} = \frac{sk_z}{k}$$

[The linear time scale is $O(Ro)$ and the nonlinear time scale for exact resonances is $O(1)$.]
The limit $Ro \to 0$

- At first-order in $Ro$, 3D fast waves and 2D slow modes ($k_z = 0$) decouple (Greenspan 1969, Waleffe 1993)
- Resonant triads cannot transfer energy from fast waves to slow 2D modes
- $128^3$ simulations of homogeneous flow at $Ro = 6 \times 10^{-4}$ with forcing at intermediate wavenumbers (Chen, Chen, Eyink & Holm 2004) an inverse cascade among 2D modes $E(k) \propto k^{-5/3}$
- no symmetry breaking
Simulations of homogeneous flow with forcing at intermediate scales (Hossain 1994 in $32^3$, Smith & Waleffe 1999 in $128^3$)

- energy transfer from 3D to 2D modes in the form of vortical columns
  
  \[ E(k) \propto k^{-3} \]

- symmetry breaking in favor of cyclones

- Our question: are near-resonances responsible?

\[ \left| \sigma_{s_k} + \sigma_{s_p} + \sigma_{s_q} \right| = O(Ro) \]
White-noise force with gaussian correlation function

\[ F(k) = \epsilon_f \frac{\exp[-0.5(k - k_f)^2/\sigma^2]}{(2\pi)^{1/2}\sigma} \]

\[ k_f = 24, \quad \epsilon_f \approx 1, \quad \sigma = 1 \]

Hyperviscosity

\[ -\nu_H \nabla^2 p_H, \quad p_H = 8 \]

Rossby number

\[ Ro = \frac{(\epsilon_f k_f^2)^{1/3}}{2\Omega} \]
Spectra from the $128^3$ simulation

Figure 2: Solid: $E(k)$; Dashed: $E(k_h, k_z = 0)$; Red line: $k^{-3}$
Cyclonic vortices in the $128^3$ simulation
Simulations of reduced models for $Ro \approx 0.1$

- Interactions among near-resonances only, with

$$|\sigma_{sk} + \sigma_{sp} + \sigma_{sq}| \leq \epsilon Ro, \quad \epsilon = O(1), \quad \sigma_{sk} = \frac{sk_z}{k}$$

- Interactions among non-resonances only, with

$$|\sigma_{sk} + \sigma_{sp} + \sigma_{sq}| > \epsilon Ro$$

- We remove triad interactions among modes, but do not remove the modes themselves.

- FFTs can no longer be used $\implies$ low resolution!
Energy spectra, $64^3$ full simulation, $Ro = 0.085$

![Energy spectra plot](image)

**Figure 3:** Black solid: $E(k)$ at $t = 69$; Black dash: $E(k_h, k_z = 0)$ at $t = 69$; Red: $E_{2d}(k_h)$ at $t = 105$
Vertical vorticity ($\hat{z}$-averaged), $64^3$ full simulation, $t = 69$
PDF of $\hat{z}$-averaged vertical vorticity, full simulation, $t = 35, 69$
There are 77, 243, 296 triads and 77, 243, 296 × 8 = 617, 946, 368 total triad interactions.

There are only 4, 688 exact 3D resonant triad interactions, and 264 with all 3 \( C_{k,p,q} \)'s nonzero.

2D triad interactions are about 0.6%. Note that 2D triad interactions are exactly resonant.

Interactions among near-resonances only, with

\[
|\sigma_{s_k} + \sigma_{s_p} + \sigma_{s_q}| \leq \epsilon \, Ro, \quad \epsilon = 1, \quad \sigma_{s_k} = s \frac{k_z}{k}
\]

are about 12% of all interactions at \( Ro = 0.085 \).
Why near-resonances?

- 3D exact resonances transfer energy toward the 2D plane

- \[ \Rightarrow \] 3D near-resonances transfer energy toward the 2D plane

- Near-2D near-resonances finish the job

- We will see that near-2D near-resonances alone are much less efficient than near-2D and 3D near-resonances together
$E(k_h, k_z = 0)$ for Near-Res. with $\epsilon = 0.3$ (3.5% of interactions)

Figure 4: Black: full; Green: $\epsilon = 0.3$; Red: $k^{-3}$
$E(k)$ for Near-Res. with $\epsilon = 0.3$ (3.5\% of interactions)

Figure 5: Black: full; Green: $\epsilon = 0.3$; Red: $k^{-3}$
Non-resonances for $\epsilon = 0.3$ (96.5\% of interactions)

Figure 6: Black: full; Blue: $\epsilon = 0.3$
Near-resonances: energy vs time for various values of $\epsilon$

Figure 7: Black: full; Green: $\epsilon = 0.3$, Blue: $\epsilon = 1.0$, Red: 2D
Spectra for $\epsilon = 1$ at $t = 52$ (12% of triads)

![Graph of spectra for $\epsilon = 1$ at $t = 52$](image)

Figure 8: Black: full; Blue: $\epsilon = 1.0$, Red: $k^{-3}$
\( \hat{z} \)-averaged vertical vorticity for \( \epsilon = 1, \ t = 52 \)
PDF of vorticity at $t = 52$: full and $\epsilon = 1$

Figure 9: Black: full; Red: $\epsilon = 1.0$
Near-resonant vs near-2d interactions

- Near-resonances include 3D near-resonances and near-2d near-resonances
- Do all near-resonances play a role, or just near-2d near-resonances?
- Define near-2d by

\[ |\sigma(k)| = \left| s \frac{k_z}{k} \right| \leq \delta \, Ro \]

E.g. \( \delta = 3 \) corresponds to a azimuthal wedge of \( 30^\circ \) about \( k_z = 0 \) (\( \theta = 15^\circ \), about 4.4% of triad interactions at \( Ro = 0.085 \)).
Figure 10: Black: full; Green: $\epsilon = 0.3$; Blue: $\delta = 3$; Red: 2D
Percentages for various wedge thicknesses, $64^3$, $Ro = 0.085$

- **Wedge of $20^\circ$ ($\theta = 10^\circ$, $\delta \approx 2$):** 2.0% of triad interactions
- **Wedge of $30^\circ$ ($\theta = 15^\circ$, $\delta \approx 3$):** 4.4%
- **Wedge of $40^\circ$ ($\theta = 20^\circ$, $\delta \approx 4$):** 8.0%
- **Wedge of $60^\circ$ ($\theta = 30^\circ$, $\delta \approx 6$):** 18.0%
- **Wedge of $80^\circ$ ($\theta = 40^\circ$, $\delta \approx 7.5$):** 27.0%
Energy evolution for various wedge thicknesses

Figure 11: Black: full; Green: 80°; Blue: 40°; Yellow: 20°; Red: 2D
Figure 12: Black: full; Green: wedge of 80°; Red: $k^{-3}$
Symmetry breaking by near-2D only, $t = 69$

Figure 13: Black: full; Green: $80^\circ$ (27% of triads)
For $\beta$-plane flow, replace $-\partial_x |\nabla|^{-1} \zeta$ by $\partial_x \nabla^{-2} \zeta$ and $Ri = U/\beta L^2$
Waves solutions in the inviscid, linear, unforced limit

\[ \zeta(x, t; k) = \hat{\zeta}(k) \exp \left[ i \left( k \cdot x - \sigma(k) \frac{t}{R \iota} \right) \right] + c.c. \]

\[ \sigma(k) = -\frac{k_x}{k}, \quad k = (k_x, k_y) \]

Normal $\beta$-plane flow has $\sigma(k) = -k_x/k^2$

Note: most other PDE models of geophysical flows have homogeneous dispersion relations with $\sigma(ak) = \sigma(k), \ a \neq 0$. 
The nonlinear solution as a superposition of waves

\[ \zeta(x, t) = \sum_{k} \hat{\zeta}(k) \exp[i(k \cdot x - \sigma(k)t)] \]

Substitution yields

\[ \left( \frac{\partial}{\partial t} + \nu k^2 \right) \hat{\zeta}(t; k) \]

\[ = \sum_{\Delta k, p, q} \mathcal{C}_{k, p, q} \hat{\zeta}^*(t; p) \hat{\zeta}^*(t; q) \exp \left[ i \left( \sigma_k + \sigma_p + \sigma_q \right) \frac{t}{Ri} \right] \]

\[ \mathcal{C}_{k, p, q} = (q^{-2} - p^{-2})(p \times q) \cdot \hat{z} \]
Reduced models for $Ri \approx 0.1$

- Interactions among near-resonances only, with

$$|\sigma(k) + \sigma(p) + \sigma(q)| \leq \epsilon Ri, \quad \epsilon = O(1), \quad \sigma(k) = -\frac{k_x}{k}$$

- Near resonances with $\epsilon = 1$ at $Ri = 0.1$

are 15% of all triad interactions.

- Interactions among non-resonances only, with

$$|\sigma(k) + \sigma(p) + \sigma(q)| \geq \epsilon Ri$$

- Low resolution is now $384^2$
Figure 14: Blue: full; Red: near-res.; Green: non-res.
Spectra

Full simulation, time = 267

Near resonances, time = 264

Non resonances, time = 268

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Streamfunction

Full, time = 267

Near resonances, time = 264

Non resonances, time = 268
Zonally averaged, zonal velocity

Full simulation, time = 267

Near resonances, time = 264
Zonally averaged, zonal velocity

Full simulation, time = 401

Near resonances, time = 398
Conclusion: Near resonances with $\epsilon = 1$ are more efficient than the full equations for generation of coherent structures and symmetry breaking.

Assertion: True also for stratified flows and rotating stratified flows with $N/f < 0.5$ and $N/f > 2$.

Comment: Models of rotating flows based on near-2d interactions with $\sigma = \pm k_z/k_h$ will likely exhibit weak symmetry breaking (Julien, Knobloch & Werne 1998, Cambon, Rubinstein & Godeferd 2004).

Challenge: To derive a computationally efficient model, which preserves resonant traces.