# Practice Exam Solutions 320



1 (a)

$$\left[\begin{array}{ccc|c} 6 & 3 & 3 & b_1 \\ 2 & 5 & -1 & b_2 \\ -4 & -8 & 1 & b_3 \end{array}\right] \quad \left[\begin{array}{ccc|c} 2 & 5 & -1 & b_2 \\ 6 & 3 & 3 & b_1 \\ -4 & -8 & 1 & b_3 \end{array}\right]$$

$$\left[\begin{array}{cccc} 2 & 5 & -1 & b_2 \\ 0 & -12 & 6 & b_1 - 3b_2 \\ 0 & 2 & -1 & b_3 + 2b_2 \end{array}\right] \quad \left[\begin{array}{cccc} 2 & 5 & -1 & b_2 \\ 0 & -12 & 6 & b_1 - 3b_2 \\ 0 & 0 & 0 & b_3 + 2b_2 + (b_1 - 3b_2)/6 \end{array}\right]$$

$$0 = b_3 + 2b_2 + b_1/6 - b_2/2 = b_3 + \frac{3}{2} b_2 + b_1/6$$

$$b_3 = -b_1/6 - \tfrac{3}{2} b_2 \qquad \underline{b} = s \begin{bmatrix} 1 \\ 0 \\ -1/6 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ -3/2 \end{bmatrix}$$

(c) $\underline{b} = \begin{bmatrix} 0 \\ 1 \\ -3/2 \end{bmatrix} \Rightarrow s = 0, \; t = 1$

$-12x_2 + 6x_3 = b_1 - 3b_2 = 0 - 3 = -3$, let $x_3 = q$

$-12x_2 = -3 - 6q$, $x_2 = \frac{1}{4} + \frac{1}{2} q$

$2x_1 + 5x_2 - q = b_2 = 1$, $2x_1 = -5\left(\frac{1}{4} + \frac{1}{2} q\right) + q + 1$

$$= -5/4 - 5/2\, q + \frac{2}{2} q + \frac{4}{4}$$

$$= -1/4 - \frac{3}{2} q$$

$x_1 = -\frac{1}{8} - \frac{3}{4}q \;,\quad x_2 = \frac{1}{4} + \frac{1}{2}q \;,\quad x_3 = q$

$$\underline{x} = \begin{bmatrix} -\frac{1}{8} \\ \frac{1}{4} \\ 0 \end{bmatrix} + q \begin{bmatrix} -\frac{3}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

b) a basis is

$$\begin{bmatrix} 1 \\ 0 \\ -\frac{1}{6} \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \\ -\frac{3}{2} \end{bmatrix}$$

③ ~~④~~

② ~~⑬~~ $4y'' + 4y' + y = 0 \quad y(4) = e^{-2} \quad y'(\cancel{4}) = 2e^{-2}$

$4r^2 + 4r + 1 = 0, \quad r = \frac{-4 \pm \sqrt{16-16}}{8} = -\frac{1}{2}$

$y = c_1 e^{-x/2} + c_2 x e^{-x/2}$

$y' = -\frac{1}{2}c_1 e^{-x/2} - \frac{1}{2}c_2 x e^{-x/2} + c_2 e^{-x/2}$

$e^{-2} = c_1 e^{-2} + 4c_2 e^{-2}, \quad 1 = c_1 + 4c_2$

$2e^{-2} = -\frac{1}{2}c_1 e^{-2} - 2c_2 e^{-2} + c_2 e^{-2}$

$2 = -\frac{1}{2}c_1 - c_2$

$\begin{bmatrix} 1 & 4 \\ -\frac{1}{2} & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \begin{bmatrix} 1 & 4 & 1 \\ -\frac{1}{2} & -1 & 2 \end{bmatrix}$

$\begin{bmatrix} 1 & 4 & 1 \\ 0 & 1 & 5/2 \end{bmatrix} \qquad c_2 = 5/2 \qquad c_1 + 4(5/2) = 1$

$c_1 = -9$

$$\boxed{y = -9e^{-x/2} + \frac{5}{2}x e^{-x/2} \qquad -\infty < x < \infty}$$

3 $\quad 4x^2y'' + y = 0 \qquad x > 0 \qquad y_1(x) = x^{1/2}$

Let $y = x^{\frac12}v$, $\quad y' = \frac12 x^{-1/2}v + x^{1/2}v'$

$$y'' = -\tfrac14 x^{-3/2}v + x^{-1/2}v' + x^{\frac12}v''$$

$$4x^2\left[-\tfrac14 x^{-3/2}v + x^{-1/2}v' + x^{\frac12}v''\right] + x^{\frac12}v = 0$$

$$\cancel{-x^{\frac12}v} + 4x^{\frac32}v' + 4x^{5/2}v'' + \cancel{x^{\frac12}v} = 0$$

$$v'' = \frac{-v'}{x}, \qquad w' = \frac{-w}{x} \qquad w = v'$$

$$\frac{dw}{w} = \frac{-1}{x}\,dx, \qquad \ln|w| = -\ln x + c_1$$

$$w = \frac{c_1}{x}, \qquad v = c_1\ln x + c_2$$

$$\boxed{y = c_1x^{\frac12}\ln x + c_2x^{\frac12}}$$

4) $c_1 \begin{bmatrix} 3 \\ 9 \\ 0 \\ 5 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 0 \\ 9 \\ -7 \end{bmatrix} + c_3 \begin{bmatrix} 4 \\ 7 \\ 5 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 3 & 3 & 4 \\ 9 & 0 & 7 \\ 0 & 9 & 5 \\ 5 & -7 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 3 & 4 & 0 \\ 9 & 0 & 7 & 0 \\ 0 & 9 & 5 & 0 \\ 5 & -7 & 0 & 0 \end{bmatrix} R_2 - 3R_1 \quad \begin{bmatrix} 3 & 3 & 4 & 0 \\ 0 & -9 & -5 & 0 \\ 0 & 9 & 5 & 0 \\ 5 & -7 & 0 & 0 \end{bmatrix}$$

switch $R_2$ and $R_4$

$$\begin{bmatrix} 3 & 3 & 4 & 0 \\ 5 & -7 & 0 & 0 \\ 0 & 9 & 5 & 0 \\ 0 & -9 & -5 & 0 \end{bmatrix} \begin{matrix} R_1/3 \\ \\ \\ R_4 + R_3 \end{matrix} \quad \begin{bmatrix} 1 & 1 & 4/3 & 0 \\ 5 & -7 & 0 & 0 \\ 0 & 9 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_2 - 5R_1 \Rightarrow$

$$\begin{bmatrix} 1 & 1 & 4/3 & 0 \\ 0 & -12 & -\frac{20}{3} & 0 \\ 0 & 9 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{l} \\ R_2/4 \\ R_3/3 \\ \\ \end{array} \quad \begin{bmatrix} 1 & 1 & 4/3 & 0 \\ 0 & -3 & -5/3 & 0 \\ 0 & 3 & 5/3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R_3 + R_2 \Rightarrow \begin{bmatrix} 1 & 1 & 4/3 & 0 \\ 0 & -3 & -5/3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Choose $C_3 \Rightarrow$

$-3\,C_2 - 5/3\,C_3 = 0$ determines $C_2$

$C_1 + C_2 + \frac{4}{3}\,C_3 = 0$ determines $C_1$

e.g. $C_3 = 1$, $C_2 = -5$, $C_1 = \frac{20}{3}$

$\Rightarrow$ linearly dependent vectors

5. $y'' + \ln(-x)\,y = 0 \qquad x < 0$

This is a linear, homogeneous, $2^{nd}$-order ODE with coefficients that are continuous for $x < 0$

The solution space is a function space of dimension 2

$\Rightarrow$ the Wronskian of 2 linearly independents solution will be

$\boxed{\text{nonzero}}$ for $x < 0$.

The statement is $\boxed{\text{FALSE}}$

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$$\underline{\underline{A}} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad \underline{\underline{B}} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

$$(\underline{\underline{A}}\,\underline{\underline{B}})^T = \left\{ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \right\}^T$$

$$= \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}^T$$

$$= \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{21}b_{11} + a_{22}b_{21} \\ a_{11}b_{12} + a_{12}b_{22} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$$

$$\underline{\underline{B}}^T \underline{\underline{A}}^T = \begin{bmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \end{bmatrix} \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}$$

$$= \begin{bmatrix} b_{11}a_{11} + b_{21}a_{12} & b_{11}a_{21} + b_{21}a_{22} \\ b_{12}a_{11} + b_{22}a_{12} & b_{12}a_{21} + b_{22}a_{22} \end{bmatrix}$$

$$\Rightarrow \left(\underline{\underline{A}}\,\underline{\underline{B}}\right)^T = \underline{\underline{B}}^T \underline{\underline{A}}^T$$ (compare the 2

and use the fact that addition & multiplication of

scalars is commutative)

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$$\underline{A} = \begin{bmatrix} 1 & 2 & -2 & 5 \\ -1 & 2 & 3 & 4 \\ 1 & 3 & 1 & -2 \\ -1 & -3 & 0 & -4 \end{bmatrix} \begin{matrix} \\ R_2 + R_1 \\ R_3 - R_1 \\ R_4 + R_1 \end{matrix}$$

$$\begin{bmatrix} 1 & 2 & -2 & 5 \\ 0 & 4 & 1 & 9 \\ 0 & 1 & 3 & -7 \\ 0 & -1 & -2 & 1 \end{bmatrix} = \underline{B} \qquad \det(\underline{B}) = \det(\underline{A})$$

$$\underline{C} = \begin{bmatrix} 1 & 2 & -2 & 5 \\ 0 & -1 & -2 & 1 \\ 0 & 1 & 3 & -7 \\ 0 & 4 & 1 & 9 \end{bmatrix} \begin{matrix} \\ \\ R_3 + R_2 \\ R_4 + 4R_2 \end{matrix} \qquad \det(\underline{C}) = -\det(\underline{A})$$

$$\underline{D} = \begin{bmatrix} 1 & 2 & -2 & 5 \\ 0 & -1 & -2 & 1 \\ 0 & 0 & 1 & -6 \\ 0 & 0 & -7 & 13 \end{bmatrix} \begin{matrix} \\ \\ \\ R_4 + 7R_3 \end{matrix} \qquad \det(\underline{D}) = -\det(\underline{A})$$

$$\underline{\underline{E}} = \begin{bmatrix} 1 & 2 & -2 & 5 \\ 0 & -1 & -2 & 1 \\ 0 & 0 & 1 & -6 \\ 0 & 0 & 0 & -29 \end{bmatrix}$$

$$\det(\underline{\underline{E}}) = -\det(\underline{\underline{A}})$$

$$\boxed{\det(\underline{\underline{A}}) = -29}$$