320 Exam 1

1. $\frac{dy}{dt} = (e^{y+2}-1)(e^{y}-1)(y-2) \qquad y(0) = y_0$

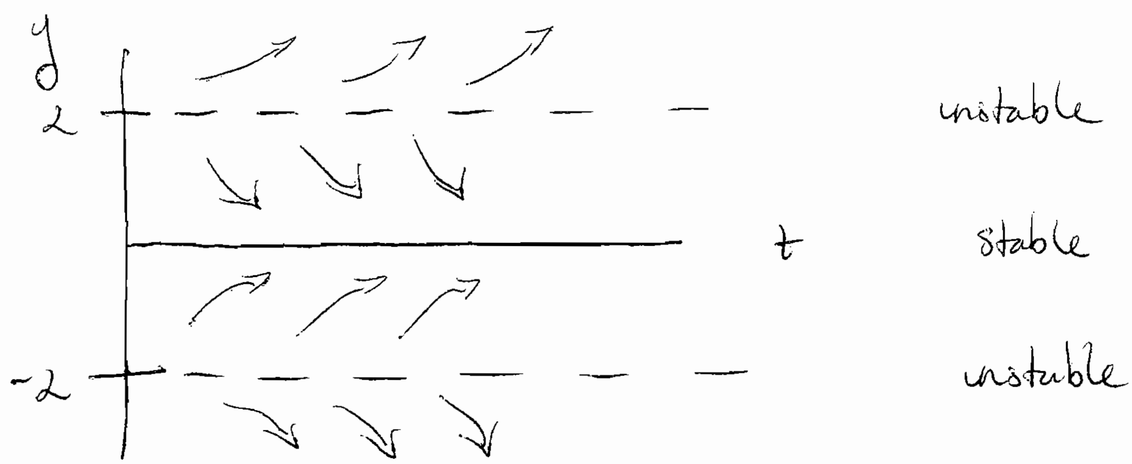
critical points: $y = -2, 0, 2$

$y < -2 \qquad \frac{dy}{dt} = - - - < 0$

$-2 < y < 0 \qquad \frac{dy}{dt} = + - - > 0$

$0 < y < 2 \qquad \frac{dy}{dt} = + + - < 0$

$y < 2 \qquad \frac{dy}{dt} = + + + > 0$



For $y_0 = -1 \qquad \lim_{t \to \infty} y(t) = 0$

2 $y' = y^3 x e^{x^2}$ $\quad y(0) = -2$

$\frac{dy}{y^3} = x e^{x^2} dx$

$-\frac{1}{2} y^{-2} = \frac{1}{2} e^{x^2} + C$, $\frac{1}{y^2} = C - e^{x^2}$

$y(0) = -2 \implies \frac{1}{4} = C - 1 = C - \frac{4}{4}$

$\implies C = \frac{5}{4}$

$\frac{1}{y^2} = \frac{5}{4} - e^{x^2} = \frac{5 - 4e^{x^2}}{4}$, $y^2 = \frac{4}{5 - 4e^{x^2}}$

$y = -\left(\frac{4}{5 - 4e^{x^2}}\right)^{1/2}$ $\quad 5 - 4e^{x^2} > 0$

$5 > 4e^{x^2}$

$\frac{5}{4} > e^{x^2}$

$\ln\left(\frac{5}{4}\right) > x^2$

$-\left[\ln\left(\frac{5}{4}\right)\right]^{1/2} < x < \left[\ln\left(5/4\right)\right]^{1/2}$

3

a $$\begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & -1 \\ -2 & 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ k \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 2 & 4 \\ 2 & 3 & -1 & k \\ -2 & 1 & -3 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 2 & 4 \\ 0 & 5 & -5 & k-8 \\ 0 & -1 & 1 & 10 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & -1 & 2 & 4 \\ 0 & 5 & -5 & k-8 \\ 0 & 0 & 0 & 10+\frac{k-8}{5} \end{bmatrix}$$

$$0x_3 = 10 + \frac{k-8}{5} = \frac{k+42}{5}$$

$k = -42$ infinite no. of solutions

$k \neq -42$ no solutions

never a unique solution

3 b $\begin{bmatrix} 1 & 0 & 3 \\ -1 & k & -1 \\ 3 & 1 & 10 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 & 3 & 8 \\ -1 & k & -1 & 4 \\ 3 & 1 & 10 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 3 & 8 \\ 0 & k & 2 & 12 \\ 0 & 1 & 1 & -24 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 3 & 8 \\ 0 & 1 & 1 & -24 \\ 0 & k & 2 & 12 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 3 & 8 \\ 0 & 1 & 1 & -24 \\ 0 & 0 & 2-k & 12+24k \end{bmatrix}$$

$$(2-k)x_3 = 12+24k = 12(1+2k)$$

$$x_3 = \frac{12(1+2k)}{2-k}$$

$k=2$ no solution

$k \neq 2$ unique solution

never infinite solutions

(4) $\frac{dy}{dx} = y + e^x \qquad y(0) = 2$

(a) Exact solution : $\frac{dy}{dx} - y = e^x$

$\mu(x) = \exp[-x]$

$\frac{d}{dx}\left[e^{-x}y\right] = 1 \quad , \quad e^{-x}y = x + C$

$y = xe^x + Ce^x \quad , \quad 2 = 0 + C$

$y = (2+x)e^x \qquad -\infty < x < \infty$

(b) One step of Forward Euler :

$y_{n+1} = y_n + hf(x_n, y_n)$

$x=h$: $y_1 = y_0 + h\left(y_0 + e^{x_0}\right)$

$y_1 = 2 + h(2+1) = 2 + 3h$

(c) One step of Modified Euler

$y_{n+1} = y_n + \frac{h}{2}\left\{f(x_n, y_n) + f(x_{n+1}, y_{n+1})\right\}$

where $y_{n+1} = y_n + hf(x_n, y_n)$

$x = h$: $y_1 = y_0 + \frac{h}{2}\{ y_0 + e^{x_0}$
$+ y_0 + h(y_0 + e^{x_0}) + e^{x_1} \}$
$= 2 + \frac{h}{2}\{2+1+2+h(2+1) + e^h\}$
$= 2 + \frac{h}{2}\{5 + 3h + e^h\}$

(d) Taylor Series at $x = h$

Exact Soln: $y(h) = (h+2)e^h$
$= (h+2)(1 + h + \frac{h^2}{2!} + \frac{h^3}{3!} + \cdots)$
$= 2 + 2h + h^2 + \frac{1}{3}h^3 + h + h^2 + \frac{h^3}{2} + O(h^4)$
$= 2 + 3h + 2h^2 + \frac{5}{6}h^3 + O(h^4)$

Forward Euler: $y(h) = 2 + 3h$

Modified Euler: $y(h) = 2 + \frac{5}{2}h + \frac{3}{2}h^2 + \frac{h}{2}e^h$
$= 2 + \frac{5}{2}h + \frac{3}{2}h^2 + \frac{h}{2}\{1 + h + \frac{h^2}{2!} + \frac{h^3}{3!} + \cdots\}$

$$= \cancel{2} + \cancel{\tfrac{5}{2}h} + \cancel{\tfrac{3}{2}h^2} + \cancel{\tfrac{h}{2}} + \cancel{\tfrac{h^2}{2}} + \frac{h^3}{4} + O(h^4)$$

$$= 2 + 3h + 2h^2 + \frac{h^3}{4} + O(h^4)$$

Taylor Series for exact soln. and Forward Euler match up to $O(h)$

$\Longrightarrow$ local truncation error of Forward Euler is $O(h^2)$

Taylor Series for exact soln. and Modified Euler match up to $O(h^2)$

$\Longrightarrow$ local truncation error of Modified Euler is $O(h^3)$