Math 320 (Smith): Practice Exam 1

1. Consider
\[ \frac{dy}{dt} = (y - a)(y^2 - b^2), \quad y(0) = -a, \quad b > a > 0. \tag{1} \]
(a) Sketch, roughly, a direction field and classify all critical points.
(b) Determine (from your sketch), the asymptotic behavior of the solution as \( t \to \infty \).

2. The following augmented coefficient matrix results from elementary row operations on a \( 3 \times 3 \) system of linear algebraic equations \( Ax = b \).
\[
\begin{pmatrix}
-1 & 1 & 1 & 2 \\
0 & 5 & -k & 4 \\
0 & 0 & k & p + 3
\end{pmatrix}
\tag{2}
\]
Consider 2 different values of the parameter \( p \): (a) \( p = -3 \), and (b) \( p = -2 \).
Determine for what values of \( k \) the system has (i) a unique solution, (ii) no solution, and (iii) infinitely many solutions.
FOR PART (a) ONLY when \( p = -3 \): Find all solutions in cases (i) and/or (iii), and write the solution \( x \) in vector form.

3. Given
\[ \frac{dy}{dx} = -\frac{y(x)}{(x - 1)} + \frac{\exp(-x)}{(x - 1)}, \quad y(0) = 2. \tag{3} \]
(a) Find the exact solution. For what values of \( x \) is the solution defined?
(b) Use one step of the Forward Euler method with step size \( h \) to find an approximation for \( y(h) \).

4. (20 points) Consider the initial value problem
\[ \frac{dy}{dx} = -\frac{5}{2}x^4 y^3, \quad y(0) = -1. \tag{4} \]
(a) Find \( y(x) \) explicitly. For what values of \( x \) is the solution defined?
(b) Use one step of the Modified Euler (Improved Euler, RK2) method with step size \( h \) to find an approximation for \( y(h) \).

5. (5 points) TRUE or FALSE: The initial value problem
\[ \frac{dy}{dt} = (y - 1)^{3/2}, \quad y(1) = 2 \tag{5} \]
is guaranteed to have a unique solution in a subrange of \( -\infty < t < \infty \).