**Practice for Final Exam**

1. Parts (a) and (b) can be solved independently. Given

$$xy' + (x^4 + 5)y = \frac{1}{x}, \quad y(-1) = -1 \tag{1}$$

(a) Find the solution and state the region where the solution exists and is unique.

(b) Find an approximate value for $y(-1+h)$ using the Forward Euler scheme with step size $h$. Write your answer in terms of $h$ only.

2. Find the general solution to the following equation:

$$2x^2y'' + 6xy' + 2y = -\frac{2}{x}, \quad x > 0. \tag{2}$$

3. Read the entire problem before starting. If you proceed efficiently, you will not need to repeat calculations. However, if you get stuck on parts (a)-(b), you should try parts (c)-(d) separately.

Given the matrix

$$\mathbf{A} = \begin{bmatrix} 2 & 3 & -1 \\ 3 & -3 & y \\ 1 & -1 & -1 \end{bmatrix} \tag{3}$$

(a) For what values of $y$ is the column space of $\mathbf{A}$ given by (i) a one-parameter subspace of $\mathbf{R}^3$, (ii) a two-parameter subspace of $\mathbf{R}^3$, (iii) $\mathbf{R}^3$?

(b) For cases (a-i) and/or (a-ii), find a basis for the column space of $\mathbf{A}$.

(c) Given $\mathbf{b}^T = [1 \;\; 3 \;\; 1]$, for what values of $y$ does $\mathbf{Ax} = \mathbf{b}$ have (i) no solution, (ii) a unique solution, (iii) an infinite number of solutions?

(d) For cases (c-ii) and/or (c-iii), find the solution $\mathbf{x}$. Write your answer in vector form.

4. Consider

$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} 2 & -4 \\ 1/4 & 4 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 48 \\ 9t \end{bmatrix}. \tag{4}$$

(a) Find the homogeneous solution.

(b) Using the method of undetermined coefficients, find a particular solution.

(c) Find the fundamental matrix.

(d) Give a formula for the solution $\mathbf{x}$ in terms of the fundamental matrix, the inverse of the fundamental matrix, the non-homogeneous term $\mathbf{g}(t)$, the initial time $t_o$ and the initial data $\mathbf{x}(t_o) = \mathbf{x}_o$. Your formula should not involve any arbitrary constants. **You do not need to find the inverse of the fundamental matrix.**

5. (a) Solve

$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} 2 & -4 \\ 1 & 2 \end{bmatrix} \mathbf{x}. \tag{5}$$

(b) Express your solution in terms of real functions.

6. Given

$$y^{(5)} - 8y^{(4)} + 16y^{(3)} + y'' - 8y' + 16y = x^2 \exp(4x) \tag{1}$$

(a) Find the homogeneous solution given that the characteristic equation is $r^5 - 8r^4 + 16r^3 + r^2 - 8r + 16 = (r^3 + 1)(r^2 - 8r + 16)$.

(b) Write down the form of the particular solution. Do not solve for the coefficients.