Final Exam Solutions Fall 2009

1. (a) $xy' + (x^4+5)y = \frac{1}{x} \quad y(-1) = -1$

$$y' + \left(x^3 + \frac{5}{x}\right)y = \frac{1}{x^2}$$

$$\mu(x) = \exp\left[\frac{x^4}{4} + 5\ln(-x)\right] = (-x)^5 \exp\left(\frac{x^4}{4}\right)$$

$$\frac{d}{dx}\left[-x^5 \exp\left(\frac{x^4}{4}\right)y\right] = \frac{-x^5}{x^2}\exp\left(\frac{x^4}{4}\right)$$

$$x^5 \exp\left(\frac{x^4}{4}\right)y = \exp\left(\frac{x^4}{4}\right) + C$$

$$y = \frac{1}{x^5} + \frac{C}{x^5}\exp\left(-\frac{x^4}{4}\right)$$

$$y(-1) = -1 - C\exp\left(-\frac{1}{4}\right) = -1$$

$$\Rightarrow C = 0$$

$$\boxed{y(x) = \frac{1}{x^5}, \quad x < 0}$$

1. b) FE $\quad y_{n+1} = y_n + hF(x_n, y_n) \quad n \geq 0$

$$y(-1+h) \cong \hat{y}_1 = y_0 + hF(x_0, y_0)$$

$$F(x,y) = \frac{1}{x^2} - \left(x^3 + \frac{5}{x}\right)y$$

$$F(x_0, y_0) = 1 - (-1-5)(-1)$$

$$= 1 + (-1-5) = -5$$

$$\boxed{y(-1+h) \cong -1 - 5h}$$

2) $2x^2y'' + 6xy' + 2y = -\frac{2}{x}$, $x > 0$

$y = Ax^r \implies 2r(r-1) + 6r + 2 = 0 = 2r^2 + 4r + 2$
$= r^2 + 2r + 1 = (r+1)^2$ +3

$y_h = C_1 x^{-1} + C_2 x^{-1} \ln x$ +3

$$\begin{bmatrix} \frac{1}{x} & \frac{\ln x}{x} \\ -\frac{1}{x^2} & \frac{1}{x^2} - \frac{\ln x}{x^2} \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{1}{x^3} \end{bmatrix} \quad R_1 \cdot \frac{1}{x}$$

+4

$$\begin{bmatrix} \frac{1}{x^2} & \frac{\ln x}{x^2} & 0 \\ -\frac{1}{x^2} & \frac{1-\ln x}{x^2} & -\frac{1}{x^3} \end{bmatrix} R_2 + R_1$$

$$\begin{bmatrix} \frac{1}{x^2} & \frac{\ln x}{x^2} & 0 \\ 0 & \frac{1}{x^2} & -\frac{1}{x^3} \end{bmatrix}$$

$\frac{u_2'}{x^2} = -\frac{1}{x^3}$

$u_2' = -\frac{1}{x}$

$$u_2 = -\ln x + C_2$$

$$\frac{u_1'}{x^2} + \frac{\ln x}{x^2}\left(-\frac{1}{x}\right) = 0$$

+4

$$u_1' = \frac{\ln x}{x} \quad , \quad u_1 = \frac{1}{2}(\ln x)^2 + C_1$$

$$y = u_1 y_1 + u_2 y_2$$

$$= \left(\frac{1}{2}(\ln x)^2 + C_1\right)\frac{1}{x} + \left(-\ln x + C_2\right)\frac{\ln x}{x}$$

$$= \frac{1}{2}\frac{(\ln x)^2}{x} + \frac{C_1}{x} - \frac{(\ln x)^2}{x} + C_2\frac{\ln x}{x}$$

+2

$$\boxed{y = -\frac{1}{2}\frac{(\ln x)^2}{x} + \frac{C_1}{x} + C_2\frac{\ln x}{x}}$$

3 $\begin{bmatrix} 2 & 3 & -1 & b_1 \\ 3 & -3 & y & b_2 \\ 1 & -1 & -1 & b_3 \end{bmatrix}$

$\begin{bmatrix} 1 & -1 & -1 & b_3 \\ 2 & 3 & -1 & b_1 \\ 3 & -3 & y & b_2 \end{bmatrix} \begin{matrix} \\ R_2 - 2R_1 \\ R_3 - 3R_1 \end{matrix}$

$\begin{bmatrix} 1 & -1 & -1 & b_3 \\ 0 & 5 & 1 & b_1 - 2b_3 \\ 0 & 0 & y+3 & b_2 - 3b_3 \end{bmatrix}$

Last raw $(y+3)\,x_3 = b_2 - 3b_3$

If $y = -3 \Rightarrow b_2 = 3b_3$ or

there is an inconsistency

If $y \neq -3 \Rightarrow$ no constraint on $\underline{b}$

(a) * $y = -3$ the column space of $\underline{\underline{A}}$ is a

2-parameter subspace of $\mathbb{R}^3$

* $y \neq -3$ the column space of $\underline{\underline{A}}$ is $\mathbb{R}^3$

* the column space of $\underline{\underline{A}}$ is never a

1-parameter subspace of $\mathbb{R}^3$

(b) $y = -3$ Find the basis vectors

$b_2 = 3b_3 = 3s \qquad b_1 = t$

$$\underline{b} = \begin{bmatrix} t \\ 3s \\ s \end{bmatrix} = t\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}$$

basis vectors $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}$

(c) $\underset{\sim}{b} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$

* Never no solution

* unique soln. for $y \neq -3$

* ∞ # solns. for $y = -3$

(d) $y \neq -3$

$$\left[\begin{array}{cccc} 1 & -1 & -1 & 1 \\ 0 & 5 & 1 & -1 \\ 0 & 0 & y+3 & 0 \end{array}\right]$$

$(y+3)x_3 = 0 \implies x_3 = 0$

$5x_2 + x_3 = -1 \implies x_2 = -1/5$

$x_1 - x_2 - x_3 = 1 \implies x_1 + \frac{1}{5} = 1$

$x_1 = \frac{4}{5}$

$$\underline{x} = \begin{bmatrix} 4/5 \\ -\frac{1}{5} \\ 0 \end{bmatrix}$$

$$y = 3 \quad \begin{bmatrix} 1 & -1 & -1 & 1 \\ 0 & 5 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_3 = p$, $\quad 5x_2 + p = -1$, $\quad x_2 = -\frac{1}{5} - \frac{p}{5}$

$x_1 - x_2 - p = 1$, $\quad x_1 = 1 + p - \frac{1}{5} - \frac{p}{5}$

$$= \frac{4}{5} + \frac{4p}{5}$$

$$\underline{X} = \begin{bmatrix} 4/5 \\ -\frac{1}{5} \\ 0 \end{bmatrix} + p \begin{bmatrix} 4/5 \\ -\frac{1}{5} \\ 1 \end{bmatrix}$$

(4) $\underline{x}' = \begin{bmatrix} 2 & -4 \\ \frac{1}{4} & 4 \end{bmatrix} \underline{x} + \begin{bmatrix} 48 \\ 9t \end{bmatrix}$

(a) $(2-\lambda)(4-\lambda) + 1 = 0, \quad 8 - 6\lambda + \lambda^2 + 1 = 0$

$\lambda^2 - 6\lambda + 9 = (\lambda - 3)^2 = 0 \qquad \lambda = 3$

$\begin{bmatrix} -1 & -4 \\ \frac{1}{4} & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \text{e.g. } \underline{v} = \begin{bmatrix} -4 \\ 1 \end{bmatrix}$

Then $(\underline{A} - \lambda \underline{\underline{I}})\, \underline{x} = \underline{v}$

$\begin{bmatrix} -1 & -4 & -4 \\ \frac{1}{4} & 1 & 1 \end{bmatrix} R_2 \cdot 4 \qquad \begin{bmatrix} -1 & -4 & -4 \\ 1 & 4 & 4 \end{bmatrix} R_2 + R_1$

$\begin{bmatrix} -1 & -4 & -4 \\ 0 & 0 & 0 \end{bmatrix} \qquad -x_1 - 4x_2 = -4$

e.g. $\underline{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$\underline{x}_h = c_1 \begin{bmatrix} -4 \\ 1 \end{bmatrix} e^{3t} + c_2 \left\{ \begin{bmatrix} -4 \\ 1 \end{bmatrix} t e^{3t} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{3t} \right\}$$

(b) Let $\underline{x}_p = \underline{a} + \underline{b}t$

$$\underline{x}' = \underline{\underline{A}}\,\underline{x} + \underline{g}_1 + \underline{g}_2 t$$

$$\underline{\underline{A}} = \begin{bmatrix} 2 & -4 \\ \frac{1}{4} & 4 \end{bmatrix} \quad \underline{g}_1 = \begin{bmatrix} 48 \\ 0 \end{bmatrix} \quad \underline{g}_2 = \begin{bmatrix} 0 \\ 9 \end{bmatrix}$$

$$\underline{b} = \underline{\underline{A}} \left\{ \underline{a} + \underline{b}t \right\} + \underline{g}_1 + \underline{g}_2 t$$

$$\underline{b} = \underline{\underline{A}}\,\underline{a} + \underline{g}_1 , \quad \underline{0} = \underline{\underline{A}}\,\underline{b} + \underline{g}_2$$

Solve for $\underline{b}$ first

$$\begin{bmatrix} 2 & -4 \\ \frac{1}{4} & 4 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -9 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 0 \\ 1 & 16 & -36 \end{bmatrix} R_2 - R_1 \quad \begin{bmatrix} 1 & -2 & 0 \\ 0 & 18 & -36 \end{bmatrix}$$

$18b_2 = -36$, $b_2 = -2$, $b_1 - 2b_2 = 0$

$b_1 = 2(-2) = -4$ $\quad \underline{b} = \begin{bmatrix} -4 \\ -2 \end{bmatrix}$

Now solve for $\underline{a}$: $\quad \underline{\underline{A}}\,\underline{a} = \underline{b} - \underline{g}_1$

$$\begin{bmatrix} 2 & -4 \\ \frac{1}{4} & 4 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} -4 \\ -2 \end{bmatrix} - \begin{bmatrix} 48 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & -26 \\ 1 & 16 & -8 \end{bmatrix} R_2 - R_1 \quad \begin{bmatrix} 1 & -2 & -26 \\ 0 & 18 & 18 \end{bmatrix}$$

$a_2 = 1$, $a_1 - 2a_2 = -26$, $a_1 = -26 + 2 = -24$

$$\underline{a} = \begin{bmatrix} -24 \\ 1 \end{bmatrix}$$

b) $\underline{x}_p = \begin{bmatrix} -24 \\ 1 \end{bmatrix} + \begin{bmatrix} -4 \\ -2 \end{bmatrix} t$

All together

$$\underline{x} = c_1 \begin{bmatrix} -4 \\ 1 \end{bmatrix} e^{3t} + c_2 \left\{ \begin{bmatrix} -4 \\ 1 \end{bmatrix} t e^{3t} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{3t} \right\}$$

$$+ \begin{bmatrix} -24 \\ 1 \end{bmatrix} + \begin{bmatrix} -4 \\ -2 \end{bmatrix} t$$

c) Fundemental matrix

$$\underline{\underline{\psi}} = \begin{bmatrix} -4e^{3t} & -4te^{3t} \\ e^{3t} & te^{3t} + e^{3t} \end{bmatrix}$$

(d) $$\underline{x}(t) = \underline{\underline{\Psi}}(t) \int_{t_0}^{t} \underline{\underline{\Psi}}^{-1}(s)\, \underline{g}(s)\, ds + \underline{\underline{\Psi}}(t)\, \underline{c}$$

$$\underline{x}(t_0) = \underline{x}_0 = \underline{\underline{\Psi}}(t_0)\, \underline{c}$$

$$\Rightarrow \quad \underline{c} = \underline{\underline{\Psi}}^{-1}(t_0)\, \underline{x}_0$$

$$\underline{x}(t) = \underline{\underline{\Psi}}(t) \int_{t_0}^{t} \underline{\underline{\Psi}}^{-1}(s)\, \underline{g}(s)\, ds + \underline{\underline{\Psi}}(t)\, \underline{\underline{\Psi}}^{-1}(t_0)\, \underline{x}_0$$

5 $\underline{x}' = \begin{bmatrix} 2 & -4 \\ 1 & 2 \end{bmatrix} \underline{x}$

$(2-\lambda)(2-\lambda)+4 = 0 = 4-4\lambda+\lambda^2+4$
$= \lambda^2-4\lambda+8$

$\lambda = \dfrac{4 \pm \sqrt{16-32}}{2} = 2 \pm 2i$

$\boxed{\lambda = 2+2i}$ $\begin{bmatrix} 2-2-2i & -4 \\ 1 & 2-2-2i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} -2i & -4 & 0 \\ 1 & -2i & 0 \end{bmatrix}$ $\begin{bmatrix} 1 & -2i & 0 \\ -2i & -4 & 0 \end{bmatrix} R_2 + 2iR_1$

$\begin{bmatrix} 1 & -2i & 0 \\ 0 & -4+4 & 0 \end{bmatrix}$ $\begin{bmatrix} 1 & -2i & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$v_1 - 2iv_2 = 0$ e.g. $\underline{v} = \begin{bmatrix} 2i \\ 1 \end{bmatrix}$

$$\underline{x} = c_1 \begin{bmatrix} 2i \\ 1 \end{bmatrix} e^{(2+2i)t} + c_2 \begin{bmatrix} -2i \\ 1 \end{bmatrix} e^{(2-2i)t}$$

(b) Write in terms of real basis

Let $\underline{x}^{(1)} = \begin{bmatrix} 2i \\ 1 \end{bmatrix} e^{(2+2i)t}$

$$= \begin{bmatrix} 2i \\ 1 \end{bmatrix} e^{2t} \{ \cos 2t + i \sin 2t \}$$

$$= \begin{bmatrix} 2ie^{2t}\cos 2t - 2e^{2t}\sin 2t \\ e^{2t}\cos 2t + ie^{2t}\sin 2t \end{bmatrix}$$

$$\mathrm{Re}\,(\underline{x}^{(1)}) = \begin{bmatrix} -2e^{2t}\sin 2t \\ e^{2t}\cos 2t \end{bmatrix}$$

$$\operatorname{Im}(\underline{x}^{(1)}) = \begin{bmatrix} 2e^{2t}\cos 2t \\ e^{2t}\sin 2t \end{bmatrix}$$

$$\underline{x} = c_3 \begin{bmatrix} -2e^{2t}\sin 2t \\ e^{2t}\cos 2t \end{bmatrix} + c_4 \begin{bmatrix} 2e^{2t}\cos 2t \\ e^{2t}\sin 2t \end{bmatrix}$$

$$= c_3 \begin{bmatrix} -2\sin 2t \\ \cos 2t \end{bmatrix} e^{2t} + c_4 \begin{bmatrix} 2\cos 2t \\ \sin 2t \end{bmatrix} e^{2t}$$

⑥ $y^{(5)} - 8y^{(4)} + 16y^{(3)} + y'' - 8y' + 16y = x^2 e^{4x}$

ⓐ $r^5 - 8r^4 + 16r^3 + r^2 - 8r + 16 = (r^3+1)(r^2 - 8r + 16)$

$= (r^3+1)(r-4)^2$

$r^3 = -1$, $(-1) = \exp[\pi i + 2m\pi i]$

$m=0$

$(-1)^{1/3} = e^{i\pi/3} = \cos \pi/3 + i \sin \pi/3 = \frac{1}{2} + \frac{i\sqrt{3}}{2}$

$m=1$

$(-1)^{1/3} = e^{i\pi} = \cos \pi + i \sin \pi = -1$

$m=2$

$(-1)^{1/3} = e^{5\pi i/3} = \cos \frac{5\pi}{3} + i \sin \frac{5\pi}{3} = \frac{1}{2} - \frac{i\sqrt{3}}{2}$

$y_h = C_1 e^{-x} + C_2 e^{\frac{x}{2}} \cos \frac{\sqrt{3}}{2} x + C_3 e^{x/2} \sin \frac{\sqrt{3}}{2} x$

$+ C_4 e^{4x} + C_5 x e^{4x}$

ⓑ $y_p = (Ax^2 + Bx + C) e^{4x} x^2$