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FOURTH GRADERS INVENT WAYS OF COMPUTING AVERAGES

Children easily learn the conventional algorithm of averaging: to find the *average* of a set of numbers, such as 80, 90, 94, and 100, just add the four numbers and divide the total by four. However, the authors are working on an approach to mathematics based on Piaget's theory, constructivism. For reasons explained later in this article and elsewhere (Kamii 1985, 1989, 1994), our approach is to encourage children to do their own thinking and to refrain from giving rules made by someone else. The purpose of this article is to describe what fourth graders can do when they are encouraged to invent their own ways of getting the average. The article also shows the teacher's active role in constructivist teaching.

Three Methods Invented by Children

We had seen or heard about three ways invented by children to compute averages. All these ways were observed in classrooms of teachers who did not teach any conventional algorithms and, instead, encouraged students to invent their own procedures. None of the children had received any instruction in how to compute averages, but they had all heard of "batting averages" and knew that the letter grades they received were "averages" of many numbers. The children may well have heard that the teacher added many scores and divided the total by the number of scores, but it is impossible to know what the children might have made of this verbal explanation.

FIGURE 1 The criteria for letter grades

A	90 – 100
B	80 – 89
C	70 – 79
D	60 – 69
F	0 – 59

The first method. The first example was invented by Nick, a third grader who had gotten 40 points on a test out of a maximum of 100. His teacher announced that the students' grade for the week would be the average of this test and the next one. Nick raised his hand and asked if he could still get a C if he made 100 on the next test. "What do *you* think?" the teacher inquired back, and Nick surprised her with a brilliant answer: "I think my average would be 70, and that's a C (see the criteria in **fig. 1**) because half of 40 is 20 and half of 100 is 50. That's a C 'cause 20 plus 50 is 70," he explained.

Nick's reasoning was difficult for us to understand. We finally figured out that it worked for the following reason: If

$$a = 40$$

and

$$b = 100$$

then

$$\frac{1}{2}a + \frac{1}{2}b = \frac{a+b}{2},$$

which is the conventional algorithm. We also tried other sets of numbers, such as 80, 90, 94, and 100. The average could be obtained either by adding the results of 80 ÷ 4, 90 ÷ 4, 94 ÷ 4, and 100 ÷ 4 or by using the conventional algorithm.

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The authors gratefully acknowledge the assistance of Barbara J. Reys, who read an earlier version of this article and offered many valuable suggestions.

Both approaches yield the same outcome: If

$$a = 80,$$
$$b = 90,$$
$$c = 94,$$

and

$$d = 100,$$

then

$$\frac{1}{4}a + \frac{1}{4}b + \frac{1}{4}c + \frac{1}{4}d = \frac{a + b + c + d}{4}.$$

The second method. The second example was furnished by Andy, a fourth grader, when his teacher introduced averaging by asking the class for the average of three bowling scores—150, 125, and 200. Estimating that the average should be a little over 150, Andy worked around this number and wrote as shown in **figure 2.** He explained that he used 25 of the 50 points that he had taken away from the 200 "to give 25 to the 125, to make it 150." After thus making three 150s, he was left with 25 points to distribute among the three scores. He first gave 5 points to each score and was left with 10 more points to distribute. That was enough to give 3 more points to each score, and the average came out to be 150 + 5 + 3 = 158, with a remainder of 1 point.

Andy's procedure baffled us, and we could not follow his reasoning even after he explained it three times! But Andy was sure of what he was doing and was willing to repeat his explanation as many times as we requested with many other sets of numbers. Eventually, we understood that Andy was making an estimate, equalizing all the scores, and then distributing the leftover points.

The third method. The third example was invented by another fourth grader in response to a homework assignment given by one of us: Think of a set of numbers you might want to average, and find the average of those numbers. The child explained that the average of 2, 9, 3, and 6 is 5 because the midpoint between 2 and 6 is 4 (see **fig. 3**), and the midpoint between 9 and 3 is 6. Since the midpoint between 4 and 6 is 5, the average of the four numbers is 5, he asserted.

We were impressed by this method of averaging two pairs of numbers first and then averaging the two averages. As another child pointed out, however, this procedure works only with sets of four numbers and is, therefore, of limited utility. (We later realized that this procedure could also work with many other sets of numbers, such as a set of eight numbers.)

FIGURE 2

Andy's way of averaging 150, 125, and 200 and getting the answer of 158 with a remainder of 1

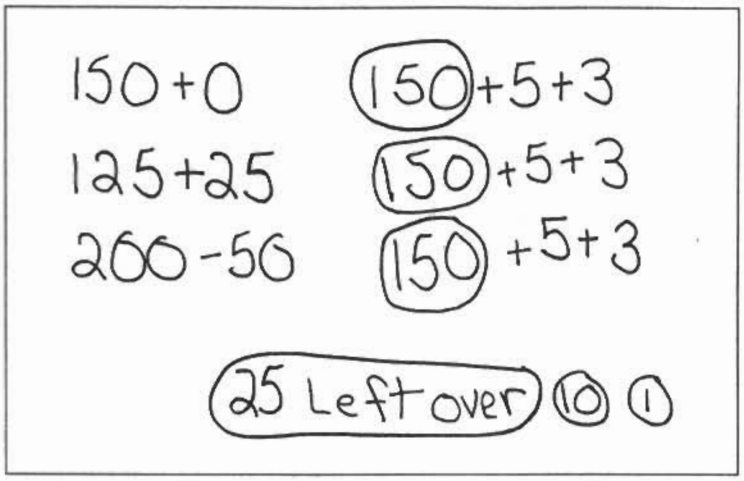
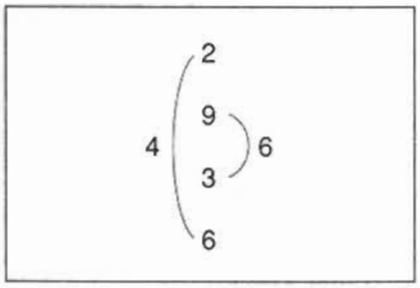


FIGURE 3

A fourth grader's procedure for getting the average of 2, 9, 3, and 6



On hearing about the preceding child-invented procedures, one of us took her turn to experiment in her class.

The first day

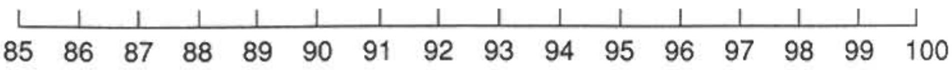
We began by asking the children what "average" meant to them. The following are some of the definitions they gave: "It's not the best and not the worst, but in the middle"; "It's a bunch of numbers put together into one"; "It's the usual amount"; and "If you get an A and an F, your average is a C." We realized how hard it was to define this term without using big words like "typical" and "representative." The children's definitions were neither general nor complete, but the examples they gave were appropriate and reflected their intuitive understanding of average based on life experiences.

The problem we planned for the first day asked for the average of three test scores—85, 95, and 100. No one invented a defensible procedure for

Children think differently from the conventional algorithm

Most of the children made good estimates

FIGURE 4 The line drawn by some children to get the average of 85, 95, and 100



getting an exact value, but the good news was that the children were able to make good estimates. The various lines of reasoning that came out during the whole-class discussion are grouped into three categories, as follows:

Qualitative reasoning using letter grades. All the children thought that for two A's and one B, according to the criteria listed in **figure 1,** the average should be "something like a low A, maybe 91."

Spatial reasoning. Some children drew the line shown in **figure 4** to represent all the numbers from 85 to 100 and said that the average had to be in the middle, around 92 or 93.

Numerical reasoning. Having decided that the average had to be between 85 and 100 and a little above 90, many children tried a variety of numerical methods in an attempt to produce a number close to 90. The following two examples are both incorrect but contain elements of correctness.

Laura wrote:

$$\begin{array}{rl} 10 & \text{(from 85 to 95)} \\ \underline{+\ 5} & \text{(from 95 to 100)} \\ 15 & \end{array}$$

Half of 15 is 8. So the average is 85 + 8 = 93.

Chris wrote the following, but Lynn disagreed with him:

$$\begin{array}{l} 100 - 95 = 5 \\ 85 + 5 = 90 \end{array}$$

The average had to be higher than 90, Lynn argued, because the score of 95 had to count for one-third of the grade. The idea of one-third indicated her awareness of an important element of averaging.

We thus learned on the first day that most of the children made good estimates and were interested enough to think hard and to struggle. We decided that the next day should begin with Nick's problem, which had only two scores, and that the second problem should involve three scores—80, 80, and 98.

The second day

The average of 40 and 100 proved to be very easy. When this problem was presented, the hands started to go up immediately, without even touching a pencil. Almost all the hands were eventually up, and the children described three similar ways of approaching the problem.

The first way was to work from both ends toward the middle by adding 10 and subtracting 10 many times (40 + 10 = 50, 100 – 10 = 90; 50 + 10 = 60, 90 – 10 = 80; and 60 + 10 = 70, 80 – 10 = 70). The second way was to take the difference between 100 and 40, divide it by 2 (60 ÷ 2 = 30), and add the result to 40 (40 + 30 = 70). The third way was a guess-and-check approach, trying to add a number to 40 and to subtract the same number from 100. Interestingly, no one in the class came up with Nick's method.

The second problem, getting the average of 80, 80, and 98, was much more difficult for the class. The children all computed the difference between 80 and 98 first but then proceeded in different ways. The most common method was this technique:

$$\begin{array}{l} 18 \div 2 = 9 \\ 80 + 9 = 89 \end{array}$$

When asked if this method made each of the three tests count equally as a third of the grade, some students divided 9 by 2, thereby getting a fourth of the difference between 80 and 98. Fortunately, these students had enough number sense to then say that they were stuck because their way did not make sense.

One child, Carl, however, did the following and was unbearably puzzled:

$$\begin{array}{l} 18 \div 3 = 6 \\ 80 + 6 = 86 \\ 80 + 6 = 86 \\ 98 - 6 = 92 \end{array}$$

We unfortunately had to stop the discussion at this point because it was time to go to physical education. The students were asked to think about Carl's idea because we would come back to this very point the next day. All the way to the gymnasium, Carl kept insisting, "I just *know* it's 86; it just *has* to be 86."

The second day thus resulted in our learning that finding the average of two numbers was very easy. Our task in preparation for the third day was to figure out how to facilitate children's use of what they knew to solve a problem with three scores.

Two things seemed to be in need of change. First, the students were focusing only on the difference between the scores, that is, between 80 and 98, whereas they needed to pay attention to the scores themselves—from 0 to 80 twice and from 0 to 98. Second, all the children except Carl had to think about how to average three scores instead of two. We decided that since some children seemed to be using lines to represent the sequence of numbers, one way to decenter their thinking was with the lines shown in **figures 5** and **6. Figures 5a** and **6** represent individual scores, and **figure 5b** shows spatially what the average of 40 and 100 means.

As a result of the preceding reflection, we also thought about a better way to define *average:* The average is a number we imagine that gives an overall picture of all the scores, a number you get by balancing out the higher scores against the lower ones.

The third day

The review of the previous day's work with **figures 5** and **6** greatly helped the children's thinking. Almost all the students thought about getting the average of 80, 80, and 98 by dividing 18 by 3, obtaining 6, and adding 6 to 80. Some children even remarked, "I know what Carl did wrong yesterday." Carl was unfortunately out of the room for a chess tournament.

The new problem for the day was to get the average of 100, 91, and 94. After drawing the three lines illustrated in **figure 7a,** the teacher asked what the person *at least made* on all three tests—the minimum that all three tests had in common. On hearing the answer of 91, the teacher drew on the chalkboard the lines illustrated in **figure 7b.** Most of the children used this hint, added 3 and 9 (because 94 – 91 = 3 and 100 – 91 = 9), divided the result by 3 (12 ÷ 3 = 4), and added 4 to 91, which equaled 95. However, two children invented other procedures that made us realize that **figure 7b** was superfluous and inappropriate.

Reasoning that the average had to be higher than 94, Lynn decided to find out if 95 was a good estimate. She took 5 points from 100 (since 100 – 5 = 95) and found out with pleasure that these points could be given to the two other scores (91 + 4 and 94 + 1) to equalize the three scores.

FIGURE 5
The lines drawn by the teacher to explain (a) the scores of 40 and 100 and (b) the average of the two scores

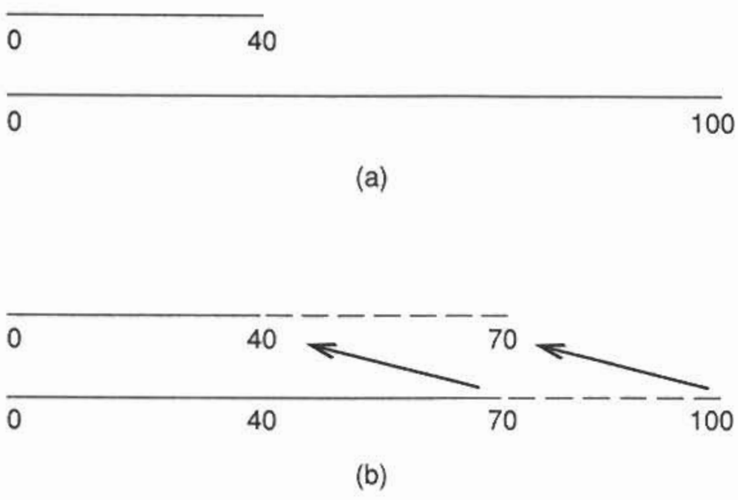
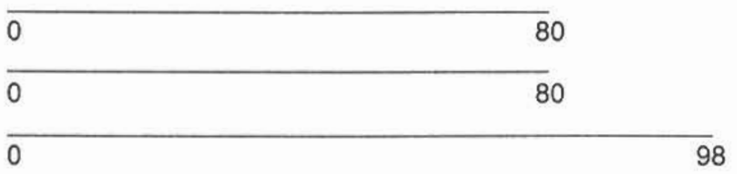


FIGURE 6
The lines drawn by the teacher to clarify the problem of getting the average of 80, 80, and 98



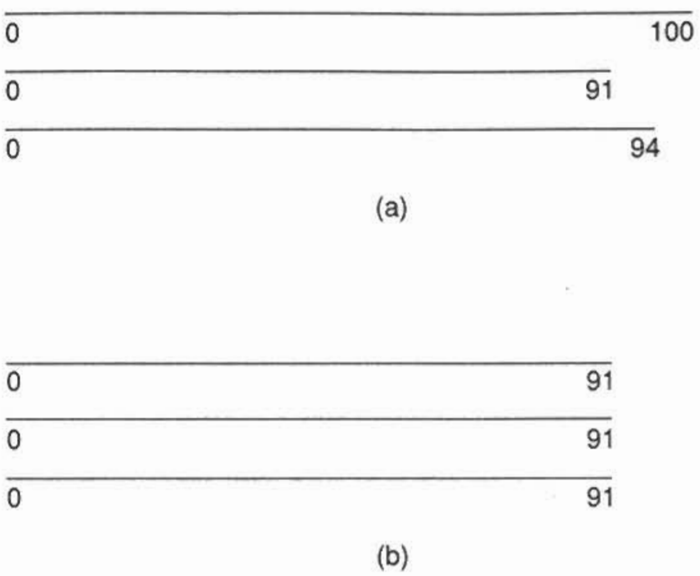
The other unexpected procedure was to try to equalize 91 and 94 first. Chris changed 94 to 93 and 91 to 92 by taking 1 away from 94 and giving it to the 91. He then took 7 from 100 (because 100 – 7 = 93) and gave 1 of the 7 to the 92 to have 93, 93, and 93. He was left with a remainder of 6, distributed the 6 points among the three scores, and ended up with an average of 93 + 2.

Once the children knew that averaging meant to equalize the scores by distributing the differences among them, other problems became easy for most of them. When three widely different scores, such as 100, 50, and 62, were later presented, some children solved them by the methods that follow. The first method was based on an estimated average of 70, but the second one began with 62, which is the score in the middle (the *median*).

"I just *know* it's 86; it just *has* to be 86"

FIGURE 7

The lines drawn by the teacher to clarify the problem of getting the average of 100, 91, and 94



Children use what they know to construct new knowledge

Some readers may be disturbed by such "equations" as "100 – 28 = 72 – 2 = 70." We do not correct this kind of writing because children in these situations are representing their *process* of thinking and using writing in the service of thinking. Correcting children in such a situation would distract them and interfere with their thinking. The convention of how to write equations will be easy to teach at a later date.

One procedure

$50 + 20 = 70$

$62 + \underline{8} = 70$

28

$100 - 28 = 72 - 2 = 70$

The average is 70 R. 2.

Another procedure

$50 + 12 = 62 + 8 = 70$

$62 + 0 = 62 + 8 = 70$

$100 - \underline{38} = 62 + 8 = 70$

$- 26$

The average is 70 R. 2.

Like Andy, these children both knew that whatever value they added to a low score had to come from a high score. Once the children "demystified" the idea of an average, some even asked if they could go over all their test scores in the teacher's grade book to see if they were averaged correctly.

Conclusion

Children easily learn the conventional algorithm for getting the average, or the arithmetic mean. However, researchers (Mokros and Russell 1992; Strauss and Bichler 1988; Pollatsek, Lima, and Well 1981) have documented children's and university students' inability to reason logically after being taught to use the algorithm. Mokros and Russell (1992, 29–30) even made the following statement within the perspective of how to make sense of large sets of data:

> Our work with adults also leads us to suspect that the arithmetic mean is a mathematical object of unappreciated complexity (belied by the "simple" algorithm for finding it) and that it should only be introduced relatively late in the middle grades, well after students have developed a strong foundation of the idea of representativeness and have a great deal of experience using the median, a measure that is not only extremely useful, but more easily connects to children's informal ideas about average.

We agree that the median is an easier concept for fourth graders to learn than the mean. However, we think that encouraging children to invent their own ways of getting the mean is a good way for them to clarify the idea of representativeness. We also think that this invention of ways to get the mean paves the way for children to reinvent the conventional algorithm.

We found that our fourth graders had three strengths: *(a)* They could easily figure out the average of two scores, *(b)* they could make good estimates, and *(c)* they all knew that they had to do *something* arithmetically with the differences among the scores. By analyzing children's errors and the fact that many of them came up with the idea of using lines to represent scores, we thought that letting them visualize all the scores in their entirety—instead of only the differences among them—might facilitate their thinking.

Children use what they know to construct, or invent, new knowledge. By analyzing what they know and how they think, the teacher can facilitate their construction of higher levels of reasoning. We think that children can reinvent the conventional algorithm for getting averages, but this hypothesis needs to be tested in the future.

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