A VARIANT OF THE LITTLE GROUP METHOD OF MACKEY AND WIGNER

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Abstract. Let be N a finite abelian group and H a finite group acting on N. The usual *little groups* method of Wigner and Mackey it is a procedure that describes all the irreducible representations of $G = N \rtimes H$ in terms of those of N and some subgroups of H. Here we develop a similar procedure for N an arbitrary finite group, and H any finite cyclic group. We note that in the case that N is abelian our construction agrees with the one of Wigner and Mackey.

Let[1] N be a group and H a finite cyclic group acting on N. Let X be the set of isomorphism classes of irreducible representations of N. Then we can define an action of $G = N \rtimes H$ on X as follows: Given $\rho \in X$, $g \in G$, and and $n \in N$ let $(g \cdot \rho)(n) := \rho(n^g)$.

Let $\Gamma = \{X_1, ..., X_m\}$ be a set of representatives of the orbits of X/H. Let $H_i := \mathrm{Stab}_H(X_i)$ be the stabilizer of the representation X_i under the action of H. For each $1 \leq i \leq m$ Let G_i be the subgroup of G given by $N \rtimes H_i$. Let χ_i be the character associated to the representation X_i, then χ_i can be extent to a character of G_i.

Proposition 1. *Let χ_i as above. There exists an irreducible character $\tilde{\chi_i}$ of G_i such that $\tilde{\chi_i}|_N = \chi_i$.*

Proof. Fix a representation $\rho_i \to \mathrm{GL}(\dim(\chi_i), \mathbb{C})$ affording χ_i, and let h_1 be a generator of H_i. By definition of H_i there exists $A_{h_1} \in \mathrm{GL}(\dim(\chi_i), \mathbb{C})$ such that, for all $n \in N$, $\rho_i(n^{h_1}) = \rho_i^{A_{h_1}}(n)$. Inductively we have that $\rho_i(n^{h_1^m}) = \rho_i^{A_{h_1}^m}(n)$ for all integer m. In particular, if m is the order of H_i, $\rho_i = \rho_i^{A_{h_1}^m}$. By Schur's lemma $A_{h_1}^m = \lambda I$ for some complex number λ. Let β any m-root of λ and $B_{h_1} := \beta^{-1} A_{h_1}$. Note that for all $n \in N$ we have that $\rho_i(n^{h_1}) = \rho_i^{B_{h_1}}(n)$. Moreover, the map $\phi : H_i \to \langle B_{h_1} \rangle$ sending h_1 to B_{h_1} defines a group homomorphism. Thus, by definition of ϕ we have that $\rho_i(n^h) = \rho_i^{\phi(h)}(n)$ for all $n \in N$ and all $h \in H_i$. Summarizing, we have shown that function $\tilde{\rho_i}$, defined below, is a group homomorphism.

$$\begin{array}{rcl} \tilde{\rho_i} : G_i & \to & \mathrm{GL}(\dim(\chi_i), \mathbb{C}) \\ (n, h) & \mapsto & \rho_i(n)\phi(h). \end{array}$$

Since clearly $\tilde{\rho_i}|_N = \rho_i$, $\tilde{\rho_i}$ defines an irreducible representation of G_i and its character $\tilde{\chi_i}$ is an extension of χ to G_i.

[1]For all the notation and background material see [Se].

□

Let ψ_i be an irreducible character of H_i and let us denote by $\tilde{\psi}_i$ the one dimensional character of G_i obtained by composition with the projection $G_i \to H_i$.

Lemma 2. *Let $\tilde{\chi}_i$ as in Proposition 1 and $\tilde{\psi}_i$ as above. Then, $\tilde{\chi}_i\tilde{\psi}_i$ is an irreducible character of G_i of dimension $dim(\chi_i)$.*

Proof. Since products of characters is a character we only have to verify that $\tilde{\chi}_i\tilde{\psi}_i$ is irreducible and of the right dimension. Recall that a character is irreducible if and only if its unitary. Hence, we calculate $[\tilde{\chi}_i\tilde{\psi}_i, \tilde{\chi}_i\tilde{\psi}_i]_{G_i} = [\tilde{\chi}_i, \tilde{\chi}_i\tilde{\psi}_i\overline{\tilde{\psi}_i}]_{G_i}$. Since $\tilde{\psi}_i$ is one dimensional $\tilde{\psi}_i\overline{\tilde{\psi}_i} = 1$, thus $[\tilde{\chi}_i\tilde{\psi}_i, \tilde{\chi}_i\tilde{\psi}_i]_{G_i} = [\tilde{\chi}_i, \tilde{\chi}_i]_{G_i} = 1$. Since the dimension of the product of characters is the product of the dimension of the characters the second part of the Lemma is clear. □

Finally we can state our main result.

Theorem 3. *Let χ_i be the irreducible character of N corresponding to $X_i \in \Gamma$. Let ψ_i be an irreducible character of H_i. Let θ_{i,ψ_i} the character of G defined by*

$$\theta_{i,\psi_i} = Ind_{G_i}^{G}\tilde{\chi}_i\tilde{\psi}_i$$

where $\tilde{\chi}_i\tilde{\psi}_i$ is the irreducible character obtained in Lemma 2. Then:

- *(i) θ_{i,ψ_i} is an irreducible character of G,*
- *(ii) $\theta_{i,\psi_i} = \theta_{j,\phi_j}$ if and only if $i = j$ and $\psi_i = \phi_j$,*
- *(iii) If χ is an irreducible character of G, then $\chi = \theta_{i,\psi_i}$ for some $X_i \in \Gamma$ and some ψ_i irreducible character of H_i.*

Proof. First note that G_i is a normal subgroup of G. This follows since G/N is abelian and G_i contains N. Hence, by Mackey's irreducibility criterion, we must show that $(n,h)\cdot(\tilde{\chi}_i\tilde{\psi}_i) \neq \tilde{\chi}_i\tilde{\psi}_i$ for all $(n,h) \notin G_i$. Let $(n,h) \in G$ such that $(n,h)\cdot(\tilde{\chi}_i\tilde{\psi}_i) = \tilde{\chi}_i\tilde{\psi}_i$. Since we can restrict this equality to any subgroup of G, we have that for all $m \in N$

$$\begin{aligned}
\tilde{\chi}_i((m,1)^{(n,h)})\tilde{\psi}_i((m,1)^{(n,h)}) &= \tilde{\chi}_i((m,1))\tilde{\psi}_i((m,1)),\\
\tilde{\chi}_i(((m^n)^h,1)\tilde{\psi}_i(((m^n)^h,1) &= \chi_i(m)\psi_i(1),\\
\chi_i((m^n)^h)\psi_i(1) &= \chi_i(m),\\
\chi_i((m^n)^h) &= \chi_i(m),\\
\chi_i(m^h) &= \chi_i(m).
\end{aligned}$$

In other words $h \in \mathrm{Stab}_H(X_i) = H_i$, or equivalently $(n,h) \in G_i$. Thus, if $(n,h) \notin G_i$, $(\tilde{\chi}_i\tilde{\psi}_i)$ is not fixed by (n,h) and this concludes the proof of (i).

To show (ii) we need first to calculate the restriction of θ_{i,ψ_i} to some subgroups. Let $(m,h_i)\in G_i$.

$$\theta_{i,\psi_i}(m,h_i) = \frac{1}{|G_i|}\sum_{(n,h)\in G}\tilde{\chi}_i((m,h_i)^{n,h})\tilde{\psi}_i((m,h_i)^{n,h}), \tag{1}$$

$$\theta_{i,\psi_i}(m,h_i) = \frac{1}{|G_i|}\sum_{(n,h)\in G}\tilde{\chi}_i(((n^{-1}mn^{h_i^{-1}})^h, h^{-1}h_ih))\psi_i(h^{-1}h_ih), \tag{2}$$

$$\theta_{i,\psi_i}(m,h_i) = \frac{\psi_i(h_i)}{|G_i|}\sum_{(n,h)\in G}\tilde{\chi}_i(((n^{-1}mn^{h_i^{-1}})^h, h_i)). \tag{3}$$

If we specify further to $h_i=1$, we obtain

$$\begin{aligned}
\theta_{i,\psi_i}(m,1) &= \frac{1}{|G_i|}\sum_{(n,h)\in G}\tilde{\chi}_i(((n^{-1}mn)^h,1)),\\
\theta_{i,\psi_i}(m,1) &= \frac{1}{|G_i|}\sum_{(n,h)\in G}\chi_i((n^{-1}mn)^h),\\
\theta_{i,\psi_i}(m,1) &= \frac{1}{|G_i|}\sum_{(n,h)\in G}\chi_i(m^h),\\
\theta_{i,\psi_i}(m,1) &= \frac{1}{|H_i|}\sum_{h\in H}\chi_i(m^h)
\end{aligned}$$

Since m is an arbitrary element of N we have just shown that

$$\theta_{i,\psi_i}|_N = \frac{1}{|H_i|}\sum_{h\in H} h\cdot\chi_i.$$

Notice that $h\cdot\chi_i = k\cdot\chi_i$ if and only if h and k are in the same coset of H. Therefore we can rewrite the right hand side of the above equation to obtain:

$$\theta_{i,\psi_i}|_N = \sum_{h\in H/H_i} h\cdot\chi_i. \tag{4}$$

Since characters are linearly independents, it follows from (4) that if $\theta_{i,\psi_i}=\theta_{j,\phi_j}$ then there is $h\in H$ such that $h\chi_i=\chi_j$. Thus, by definition of Γ we have that $i=j$. On the other hand let ψ_i' be a extension of ψ_i from H_i to G. Note that such a extension exists since H is a cyclic quotient of G. It follows from equation (3) that $\overline{\psi_i'}\theta_{i,\psi_i} = \overline{\phi_i'}\theta_{i,\phi_i}$ hence $\psi_i=\phi_i$.

Next, we show that the sum of the squares of the degrees of the representations θ_{i,ψ_i} is equal to $|G|$. This together with (ii) imply (iii). Note that $\deg(\theta_{i,\psi_i}) = [H:H_i]\deg(\chi_i)$. Hence,

$$\begin{aligned}
\sum_{1\leq i\leq|\Gamma|,\psi_i\in\mathrm{Irr}(H_i)}\deg^2(\theta_{i,\psi_i}) &= \sum_{i=1}^{|\Gamma|}\sum_{\psi_i\in\mathrm{Irr}(H_i)}[H:H_i]^2\deg^2(\chi_i)\\
&= |H|\sum_{i=1}^{|\Gamma|}[H:H_i]\deg^2(\chi_i)\\
&= |H|\sum_{\chi\in\mathrm{Irr}(N)}\deg^2(\chi)\\
&= |H||N|\\
&= |G|.
\end{aligned}$$

□

References

[Se] J. P. Serre, *Linear Representations of Finite Groups*, New York: Springer-Verlag, 1977.

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