

Modules

Basic Definitions

We now move on to the study of modules. Modules are generalizations of the notion of a vector space over a field. Instead, modules are defined over an arbitrary ring. Through this notes R will denote a ring with unity.

Definition. A right R -module M is an abelian group (written additively), together with an action of R on M (denoted by multiplication). This action satisfies the following.

1. For $x, y \in M$ and $r \in R$, $(x + y)r = xr + yr$.
2. For $x \in M$ and $r, s \in R$, $x(r + s) = xr + xs$ and $x(rs) = (xr)s$.
3. For $x \in M$, $x1 = x$.

Remark: There is also an analogous definition of a left module over R , instead of having the action on the right the action is on the left. All the results we state about right modules also hold for left modules so I wont bother in writing the same results twice.

Once an algebraic object is defined, natural questions to ask are how to define subobjects, quotient objects, and object homomorphisms.

Definition. If M is a right R -module, a subset $N \subseteq M$ is a submodule of M if the module operations defined on M make N into a module.

Here are some key examples of modules. If R is a ring with unity, then $M = R$ using the multiplication in R it may be thought of as a right R -module. More generally, if I is a right ideal of R , that is an abelian group such that if $i \in I$ and $r \in R$ then $ir \in I$, then I is a right R -module. The right ideals of R are precisely the right R -submodules of R . Analogously the left ideals of R are precisely the left R -submodules of R . A subset I of R which is both a right and left ideal is called a *bilateral ideal*.

When dealing with groups and rings, only certain subobjects allow one to construct a quotient object. For groups, we can only define a quotient group when the corresponding subgroup is normal, and for rings, we can only define a quotient ring when the subobject is an ideal. However, quotient modules always exist.

Definition. Suppose that M is a right R -module and N is a submodule of M . We define $M/N = \{N + x : x \in M\}$, with the addition and multiplication given by $(N + x) + (N + y) = N + (x + y)$, and $(N + x)r = N + xr$.

These operations make M/N into a right R -module.

Definition. If M and N are right R -modules, an R -module homomorphism from M to N is a map $\phi : M \rightarrow N$ satisfying $\phi(x + y) = \phi(x) + \phi(y)$ for $x, y \in M$ and $\phi(xr) = \phi(x)r$ for $r \in R$.

There are isomorphism theorems for modules analogous to the isomorphism theorems for groups. The correspondence theorem in this case states that there is a one-to-one correspondence between submodules of M/N and submodules of M containing N , formally

Theorem (Correspondence). Suppose that M is a right R -module and N is a submodule of M . Let $\phi : M \rightarrow M/N$ be the natural projection map. Let

$$\mathcal{U} = \{ \text{modules } P : N \subseteq P \subseteq M \}$$

and

$$\mathcal{V} = \{ \text{modules } Q : 0 \subseteq Q \subseteq M/N \}.$$

Then, the map $P \mapsto \phi(P)$ gives a bijection between \mathcal{U} and \mathcal{V} . Moreover, this correspondence respects containment, and quotient modules.

We also have the classical isomorphism theorems. Let M be an R -module and K, L be submodules then:

- $(K + L)/K \cong L/(K \cap L)$.
- If $K \subseteq L$ then $(M/K)/(L/K) \cong M/L$.

Hom

If M and N are right R -modules, we can consider the set of R -module homomorphisms from M to N . The potentially surprising fact is that this set will be another right R -module.

Definition. Let $\text{Hom}_R(M, N)$ denote the set of R -module homomorphisms from M to N . This set is given an R -module structure as follows. Suppose that $r \in R$ and $\phi, \psi \in \text{Hom}_R(M, N)$. Then, we define $(\phi + \psi)$ by $(\phi + \psi)(m) = \phi(m) + \psi(m)$ for all $m \in M$ and we define (ϕr) by $(\phi r)(m) = \phi(m)r$ for all $m \in M$.

In the situation when $M = N$ any homomorphism will map M to itself and hence we may compose two homomorphisms in $\text{Hom}_R(M, M)$. This operation makes $\text{Hom}_R(M, M)$ into a ring, which we denote $\text{End}_R(M)$. We call its elements (homomorphisms from M to itself) endomorphisms.

Simple Objects

Another natural object of study is an algebraic object O that has no non-trivial subobjects. In group theory, these are the simple groups.

Definition. A right R -module M is called simple if there are no R -submodules N of M with $0 < N < M$.

A ring R such that R (thought of as a right R -module) is simple is called a division ring. The next result gives another relationship between simple R -modules and division rings. To illustrate the concepts I've discussed so far, I will prove this result.

Lemma (Schur). Suppose that M is a simple right R -module. Then, $\text{End}_R(M)$ is a division ring.

Proof. Suppose that $\alpha : M \rightarrow M$ is a nonzero endomorphism.

Let $N = \ker \alpha$ and $O = \text{im} \alpha$. Clearly N and O are submodules of M . Since M is simple and $\alpha \neq 0$ we must have that $N = 0$ and $O = M$, in other words α is a bijection. Now, let $\beta = \alpha^{-1} : M \rightarrow M$. Suppose that $x, y \in M$. Then, there exist $u, v \in M$ such that $x = \alpha(u)$ and $y = \alpha(v)$. Then, $(x + y)\beta = ((u)\alpha + (v)\alpha)\beta = u + v = (x)\beta + (y)\beta$. Similarly, if $x \in M$ and $u \in M$ such that $x = \alpha(u)$ and $r \in R$ then $(xr)\beta = ((ur)\alpha)\beta = ur = (x)\beta r$, so β is a R -module homomorphism. We've proved that for every nonzero element $\alpha \in \text{End}_R(M)$ there is $\beta \in \text{End}_R(M)$ such that $\alpha\beta = \beta\alpha = 1$, i.e $\text{End}_R(M)$ is a division ring. \square

Corollary. *Let R be a ring which is simple as a right R -module. Then R is a division ring.*

Proof. The result follows immediately from the proposition since $R \cong \text{End}_R(M)$ as rings (See exercises). \square

Jacobson Radical

Given a right R -module M there is a natural map of right R -modules, that at the same time is a ring homomorphism.

$$\begin{aligned} \phi_M : R &\rightarrow \text{End}_R(M) \\ r &\mapsto \phi_M(r) \end{aligned}$$

$$\begin{aligned} \phi_M(r) : M &\rightarrow M \\ m &\mapsto mr \end{aligned}$$

The *annihilator* of M , denoted by $\text{ann}_R(M)$, is defined to be $\ker(\phi_M)$. By Schur's lemma we have that $\text{ann}_R(M)$ is a maximal right ideal of R if M is a simple module. Conversely every maximal right ideal J of R is of the form $\text{ann}_R(M)$ for some simple module M , namely R/J . All this is summarized in [Isaacs, 13.1]

Definition. *The jacobson radical, denoted by $J(R)$, is the intersection of all the maximal right ideals of R .*

There is a more explicit way to describe the elements of $J(R)$, but for this we need a definition. An element $r \in R$ is *right quasiregular* if $1 - r$ has a right inverse in R . Equivalently r is right quasiregular if the right ideal generated by $1 - r$ is R . Analogously we can define left quasiregular element. A quasiregular element is an element that is both left and right quasiregular.

Theorem (Isaacs, 13.4). *Let $x \in R$, then $x \in J(R)$ if and only if xr is right quasiregular for all $r \in R$.*

All the definitions we've just seen can be done for left R -modules. In particular there is also a notion of left Jacobson radical, $J_\ell(R)$. It turns out that there is no difference between such notions.

Theorem (Isaacs, 13.8). *$J_\ell(R) = J(R)$, in particular $J(R)$ is a bilateral ideal of R .*

Chain conditions and Composition Series

Modules are the generalization of vector spaces over arbitrary rings. The best kind of vector spaces to deal with are the finite dimensional. The natural analogous in modules are the ones with composition series. Here I'll recall some of the basic definitions of modules with some kind of chain conditions as composition series, ACC, DCC.

Definition. *If M is a right R -module, a composition series for M is a sequence of submodules*

$$0 = N_0 < N_1 < N_2 < \cdots < N_n = M,$$

with the property that N_{i+1}/N_i is a simple R -module.

While not all right R -modules will have a composition series, those that do have many nice properties.

Theorem (Jordan-Hölder). *Suppose that M is a right R -module and*

$$\mathcal{H} : 0 = H_0 < H_1 < \cdots < H_h = M,$$

and

$$\mathcal{K} : 0 = K_0 < K_1 < \cdots < K_k = M$$

are two composition series for M . Then, $h = k$ and the composition factors H_{i+1}/H_i and K_{i+1}/K_i for $0 \leq i < h$ are (up to isomorphism) just rearrangements of each other.

If there is a composition series as above, we called the integer h the *length* of M and we will denote it by $\ell(M)$.

Definition. *We say that a module M satisfies the ascending chain condition (ACC) if whenever M_1, M_2, \dots are submodules with $M_1 \leq M_2 \leq \dots$, then there is an integer n such that $M_n = M_{n+1} = \dots$.*

There is a similar definition of the descending chain condition where the inequalities are reversed.

Example: A finite dimensional K -vector space V satisfies both ACC and DCC. Also it is a simple exercise to show that $\ell(V) = \dim_K(V)$.

Definition. *If M is a module, we say that M is noetherian (or artinian) if M satisfies the ACC (or the DCC).*

A module M satisfies the maximal (or minimal) condition if every nonempty set of submodules has a maximal (or minimal) element.

Lemma (Isaacs, 11.2). *If M is a module, then M is noetherian (or artinian) if and only if the submodules of M satisfies the maximal (or minimal) condition.*

If we have a finite dimension vector space, every subspace and every quotient space are also finitely dimensional. Also if we have finitely many vector spaces of finite dimension their direct sum is also finite dimensional. In other words finite dimensionality is a property closed under sub, quotients and finite direct sums.

Theorem. *The condition noetherian (or artinian) is inherited by submodules, quotients and finite direct sums.*

Proof. ACC and DCC trivially are respected by subs. By correspondence they are also respected by quotients. The direct sums claims follows easily from the next result(See also [Isaacs, 11.8]). \square

Theorem (Isaacs, 11.6). *Let M be an R -module and N a submodule. M is noetherian (or artinian) iff N and M/N are noetherian (or artinian).*

The following result is key in the developing of theory of modules. Its proof requires Zorn's lemma, and I'll wait until the end of this notes to provide such proof. In the moment it is important to know some of its corollaries.

Proposition. *Suppose that $M > 0$ is a finitely generated right R -module. Then M contains a maximal submodule N [a maximal submodule N is a proper submodule such that the only larger submodule is M itself].*

Here are some of the most important corollaries of this proposition.

Corollary. *If M is a finitely generated right R -module and L is a proper submodule of M , then L is contained in a maximal submodule.*

Corollary. *If R is a ring and I is a right ideal of R , then I is contained in a maximal right ideal.*

We have to be careful with maximal subobjects. It seems from the practice that they always exist, but as is shown in the exercises that is not the case.

Theorem (Isaacs, 11.4). *Let M be an R -module. Then, M is noetherian if and only if every submodule of M is finitely generated.*

Example: Since \mathbb{Z} is a PID every ideal is finitely generated, hence it is a noetherian \mathbb{Z} -module.

Theorem (Isaacs, 11.3). *Let M be an R -module. Then, M has a composition series if and only if M is noetherian and artinian.*

Finally, there is sometimes a relationship between the properties of the ring R and modules M .

Definition. *If R is a ring, we say that R is right noetherian or right artinian if when R is viewed as an R -module it is right noetherian or right artinian.*

Theorem (Isaacs, 12.19). *Let R be a right noetherian [or a right artinian] ring. Let M be a finitely generated R -module. Then, M is noetherian [or artinian].*

From this result we can deduce that every subgroup of a finitely generated abelian group is finitely generated.

Zorn's Lemma

Now I'll discuss Zorn's lemma. Before we begin, we will need some definitions.

Definition. *A partially ordered set (or poset) P is a set together with a binary relation \leq on P . It satisfies the following properties.*

1. For all $a \in P$, $a \leq a$.
2. If $a, b, c \in P$ with $a \leq b$ and $b \leq c$ then $a \leq c$.
3. If $a \leq b$ and $b \leq a$ then $a = b$.

The key example of a partially ordered set is the set of all subobjects of an algebraic object. For example, if R is a ring, we might take P to be the set of all right ideals of R with the order relation defined as $A \leq B$ if $A \subseteq B$.

A number of important properties of modules and rings are related to properties of the relevant poset. Here are a number of properties a poset may have. Assume that P is a poset.

Definition. *If $S \subseteq P$ is a subset, we say that $m \in P$ is a maximal (or minimal) element of S if there is no $s \in S$ such that $s > m$ (or $s < m$). We say that P satisfies the maximal (or minimal) condition if every nonempty subset of P has a maximal (or minimal) element.*

Definition. *If $S \subseteq P$ is a subset, we say that S is linearly ordered if for any $s, t \in S$ either $s \leq t$ or $t \leq s$.*

Now, we can state Zorn's lemma, which is a key tool in constructing maximal elements of posets.

Lemma. *Let P be a nonempty poset and assume that whenever $L \subseteq P$ is a linearly ordered subset, there is an element $u \in P$ such that $u \geq x$ for every element $x \in L$. Then P contains a maximal element.*

Remark. An element $u \in P$ such that $u \geq x$ for every element $x \in L$ is called an “upper bound” for L .

Remark. Zorn’s lemma is equivalent to the Axiom of Choice. This axiom is independent from the other usual axioms of set theory ZF, but most mathematicians choose to assume it. If you are interested in this kind of topics Math-770 is a good start.

Now we are in position to provide the proof of the following proposition.

Proposition. Suppose that $M > 0$ is a finitely generated right R -module. Then M contains a maximal submodule N .

Proof. Let \mathcal{S} be the poset of proper submodules of M . We will show that \mathcal{S} is “Zornable,” that is, it satisfies the hypotheses of Zorn’s lemma. First note that $0 \in \mathcal{S}$, and so \mathcal{S} is nonempty. Let $\mathcal{L} \subseteq \mathcal{S}$ be a linearly ordered subset. Let

$$T = \bigcup_{L \in \mathcal{L}} L$$

be the union of the elements of \mathcal{L} . It is clear that $L \subseteq T$ for all $L \in \mathcal{L}$. I claim that $T \in \mathcal{S}$. If this is true, then T will be an upper bound for \mathcal{L} and hence \mathcal{S} is Zornable.

To show that, it suffices to show that T is a submodule of M and that T is proper.

Suppose that $x, y \in T$. Then, there exists $L_x \in \mathcal{L}$ and $L_y \in \mathcal{L}$ such that $x \in L_x$ and $y \in L_y$. Since \mathcal{L} is linearly ordered, $L_x \subseteq L_y$ or $L_y \subseteq L_x$. In the first case, $x + y$, xr and yr are all in L_y , and in the second case, $x + y$, xr and yr are all in L_x . Hence, $x + y$, xr , and yr are in T and hence T is a submodule of M .

Since M is finitely generated, suppose that $x_1, \dots, x_n \in M$ generate M . That is, the smallest submodule of M containing x_1, \dots, x_n is M itself. I claim that there is an i such that $x_i \notin T$ and hence T is proper.

Suppose to the contrary that $T = M$. Then x_1, \dots, x_n are all in T and there are $L_{x_1}, L_{x_2}, \dots, L_{x_n}$ in \mathcal{L} with $x_i \in L_{x_i}$.

Now, I claim that there is an $L \in \mathcal{L}$ such that $x_1, \dots, x_n \in L$. I will prove by induction that for each k , $1 \leq k \leq n$, there is an $L \in \mathcal{L}$ with $x_1, \dots, x_k \in L$. This is true for $k = 1$ since $x_1 \in L_{x_1}$.

Now, assume that $x_1, \dots, x_k \in L$. Since \mathcal{L} is linearly ordered, either $L \subseteq L_{x_{k+1}}$ or $L_{x_{k+1}} \subseteq L$. In the first case, $x_1, \dots, x_{k+1} \in L_{x_{k+1}}$ and in the second case, $x_1, \dots, x_{k+1} \in L$. The result follows by induction.

Hence, there is an $L \in \mathcal{L}$ such that $x_1, \dots, x_n \in L$. This implies that $\langle x_1, \dots, x_n \rangle = M \subseteq L$. This contradicts the assumption that L is a proper submodule of M . Hence, T is proper submodule of M so $T \in \mathcal{S}$. Thus, the hypotheses of Zorn’s lemma applies and \mathcal{S} contains a maximal element. This maximal element must be a maximal proper submodule N . \square

Problems

- Let R be a ring with 1 and let M be a right R -module, and let $\theta : M \rightarrow M$ be an R -homomorphism.
 - i) Suppose that M is noetherian. Show that θ is injective if it is surjective.
 - ii) Suppose that M is artinian. Show that θ is surjective if it is injective.
- Show that every vector space has a basis. (Hint: Show that a maximal linearly independent subset is a basis)

- Show that \mathbb{Q} with the usual addition does not have maximal subgroups.
- Suppose that F is a field. What are the isomorphism classes of simple F -modules U ? For these U , what does $\text{End}_F(U)$ look like?
- Suppose that M is a right R -module and $N \subseteq M$ is an R -module. If M has a composition series, prove that there is a composition series for M of the form

$$0 = N_0 < N_1 < \cdots < N_n = M,$$

where $N = N_i$ for some integer i .

- Suppose that M is a right R -module that has a composition series and N is a submodule of M . Prove that

$$\ell(N) + \ell(M/N) = \ell(M)$$

- Show that for any ring $J(R/J(R)) = 0$.
- Let $z \in R$ nilpotent and central. Then $z \in J(R)$. (z is central if $zr = rz$ for all r in R)
- 5Aug-91, 2Jan-92, 2Aug-93, 2Jan-94, 2Jan-97, 5Aug-97, 5Jan-99, 2Jan-00, 2Jan-01, 5Jan-02, 5Jan-03, 5Jan-04, 2Aug-04, 5Jan-05, 2Aug-05, 5Jan-06, 5Aug-07, 5Jan-08.