INTEGRAL TRACE FORMS DEFINING DIFFERENT GENUS OVER ARITHMETICALLY EQUIVALENT NUMBER FIELDS

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Abstract. In this short communication we exhibit two arithmetically equivalent number fields K and L such that their trace forms belong to different integral genus.

1. Introduction

It is known by work of R. Perlis (see [P]) that the Zeta function of a number field completely characterizes the rational genus of the trace form. It is natural then to ask, to what extend the same is true for the integral genus. The answer to this question was provided by B. Erez, J. Morales and R. Perlis in [EMP]. They showed that for tamely ramified number fields, the discriminant and the rational genus characterize the integral genus. In particular, tamely ramified number fields that are arithmetically equivalent have integral trace forms in the same genus. Here we provide two integral quadratic forms having the same rational genus but not the same integral one, furthermore they correspond to two arithmetically equivalent number fields.

2. Notation, definitions and example.

Let K be a number field and O_K be its maximal order.

Definition 1. *The trace form is the rational quadratic form defined by*

$$\begin{array}{cccc} \tilde{q}_K : & K & \to & \mathbb{Q} \\ & x & \mapsto & tr_{K/\mathbb{Q}}(x^2). \end{array}$$

The The integral trace form q_K is just the integral quadratic form obtained by restricting $\tilde{q}_K$ to O_K.

Definition 2. *Two number fields K, L are called arithmetically equivalent if their Riemann zeta functions ζ_K, ζ_L coincide.*

The following theorem of Perlis ([P1]) gives a simple tool to find out when two number fields are arithmetically equivalent.

Theorem 3. *Let K, L be number fields and let N to be the Galois closure of KL over $\mathbb{Q}$. Let $G = Gal(N/\mathbb{Q}), H = Gal(N/K)$ and $I = Gal(N/L)$. Then K and L are arithmetically equivalent if and only if $Ind_H^G(1) = Ind_I^G(1)$.*

Using this characterization, local considerations and the interpretations of the Hasse invariants of $\tilde{q}_K$ given by Serre (see [S]) Perlis showed (see [P]):

Theorem 4. *Let K and L be two arithmetically equivalent number fields, then $\tilde{q}_K$ and $\tilde{q}_L$ are in the same rational genus. In other words they are rationally equivalent.*

In the case of integral genus one extra hypothesis has to be added (see [EMP]);

Theorem 5. *Let K and L be two arithmetically equivalent number fields. Suppose that K and L are tamely ramified. Then q_K and q_L are in the same integral genus.*

The following example shows that the ramification condition can't be dropped.

Example 6.

Let K and L be the number fields defined by the polynomials $p_K := x^8 + 15$ and $p_L := x^8 + 240$ respectively. Let y be a root of p_K and z a root p_L. Then the following are integral basis for K and L respectively:

$$B_K = \{1, y, y^2, y^3, \frac{1}{2}(y^4+1), \frac{1}{2}(y^5+y), \frac{1}{4}(y^6+y^4+y^2+1), \frac{1}{8}(y^7+y^6+y^5+y^4+y^3+y^2+y+1)\}$$

$$B_L = \{1, z, \frac{1}{2}z^2, \frac{1}{4}(z^3+2z), \frac{1}{8}(z^4+4), \frac{1}{16}(z^5+4z^2+12z+8), \frac{1}{32}(z^6+2z^4+4z^2+8), \frac{1}{64}(z^7+2z^5+4z^4+12z^3+16z^2+24z+16)\}.$$

Calculating the Gramm Matrix of the trace form in these basis one gets the matrices

$$M_K = \begin{pmatrix} 8 & 0 & 0 & 0 & 4 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -15 \\ 0 & 0 & 0 & 0 & 0 & 0 & -30 & -15 \\ 0 & 0 & 0 & 0 & 0 & -60 & 0 & -15 \\ 4 & 0 & 0 & 0 & -28 & 0 & -14 & -7 \\ 0 & 0 & 0 & -60 & 0 & 0 & 0 & -15 \\ 2 & 0 & -30 & 0 & -14 & 0 & -22 & -11 \\ 1 & -15 & -15 & -15 & -7 & -15 & -11 & -13 \end{pmatrix}; \qquad M_L = \begin{pmatrix} 8 & 0 & 0 & 0 & 4 & 4 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -30 \\ 0 & 0 & 0 & 0 & 0 & 0 & -30 & 0 \\ 0 & 0 & 0 & 0 & 0 & -30 & 0 & -30 \\ 4 & 0 & 0 & 0 & -28 & 2 & -14 & -14 \\ 4 & 0 & 0 & -30 & 2 & 2 & -14 & -44 \\ 2 & 0 & -30 & 0 & -14 & -14 & -22 & -22 \\ 2 & -30 & 0 & -30 & -14 & -44 & -22 & -52 \end{pmatrix}$$

Since clearly M_L is zero as an element of $M_{2\times 2}(\mathbb{Z}/2\mathbb{Z})$ and M_K is not, the integral quadratic forms q_K and q_L, which in coordinates can be written as

$$q_K = 8x_1^2 - 28x_5^2 - 22x_7^2 - 13x_8^2 + 8x_1x_5 + 4x_1x_7 + 2x_1x_8 - 30x_2x_8 - 60x_3x_7 - 30x_3x_8 - 120x_4x_6 - 30x_4x_8 - 28x_5x_7 - 14x_5x_8 - 30x_6x_8 - 22x_7x_8$$

$$q_L = 8x_1^2 - 28x_5^2 + 2x_6^2 - 22x_7^2 - 52x_8^2 + 8x_1x_5 + 8x_1x_6 + 4x_1x_7 + 4x_1x_8 - 60x_2x_8 - 60x_3x_7 - 60x_4x_6 - 60x_4x_8 + 4x_5x_6 - 28x_5x_7 - 28x_5x_8 - 28x_6x_7 - 88x_6x_8 - 44x_7x_8$$

are not in the same integral genus. On the other hand using Theorem 3 one checks that K and L are arithmetically equivalent. In particular the above quadratic forms are rationally equivalent but not in the same rational genus.

References

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