STUDY GUIDE AND SAMPLE PROBLEMS FOR THE SECOND MIDTERM
MATH 222, SPRING 2009

LIST OF TOPICS FOR THE SECOND MIDTERM

- *Convergence of Taylor series.*
  You should be able to show that the Taylor series of a given function $f(x)$ converges to that function by showing that the remainder term goes to zero.
- *Complex numbers.*
  Understand arithmetic (how do you divide by a complex number? what happens to argument and absolute value when you multiply).
  The complex exponential: Definition. How do you derive the addition formulas for sine&cosine from Euler's formula?
  Finding real and complex roots of polynomial equations. How do you solve $z^n = a$? There may not be a direct question of this type on the test, but you need to know how to do this when you solve linear higher order diffeqs.
- *Differential equations.*
  First order separable and linear equations. Initial value problems (find "C" if you are given $y(0) = \dots$). Be prepared to answer simple questions about the solution you found, like "what is $\lim_{x\to\infty} y(x)$" (these limits will never be difficult, but they test whether your solution is in usable form).
  Higher order linear equations with constant coefficients. Know how to solve the homogeneous equation. Solve an initial value problem. The general solution of the inhomogeneous equation is the sum of any particular solution and the general solution of the homogeneous equation. Be able to find particular solutions by "educated guessing."
  Applications: you will not assume to have any special non-mathematical background knowledge, we will give you the equation to study.

THE PROBLEMS

Look for similar problems in the notes. Do not assume that the exam will be exactly like these problems. This is just a sample of what the level will be. The length of the problems does not correspond to the length of the problems in the exam.

(1) (a) Show that the Taylor series for $f(x) = \cos(3x - 2)$ converges for *all* real numbers x.
  (b) Show that the Taylor series for $g(x) = e^{x^2}$ converges for *all* real numbers x.
  (c) See also problems 238–248 from the notes.

(2) Figure 1 shows two complex numbers z and w.
  (a) Draw $(1 + 2i)z$ in the same figure.
  (b) Compute w/z.

(3) Let $z = 1 + i\sqrt{3}$.
  (a) Find the smallest integer $n > 0$ for which z^n is a negative real number.
  (b) Find the smallest integer $n > 0$ for which z^n is a positive real number.
  (c) Same as above if $z = -\sqrt{3} + i$.
  (d) Same as above if $z = -2 - 2i$.
  (e) Find all complex numbers z for which $z^2 = -i$.
  (f) Find all complex numbers z for which $z^3 = i$.
  (g) Find all complex numbers z for which $z^2 = 1 + i$.

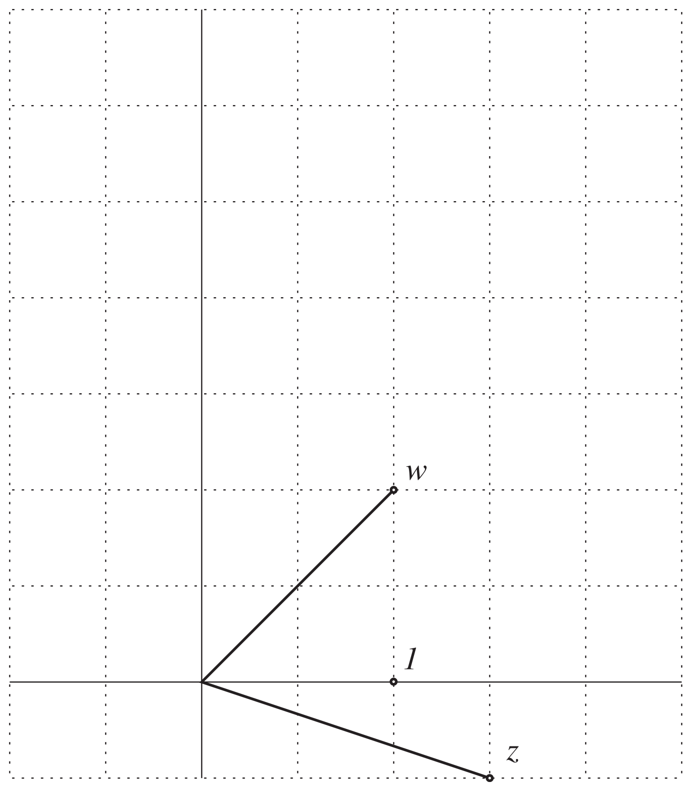

FIGURE 1. Find $(1+2i)z$ and w/z.

(4) You are given an angle x whose Cosine and Sine are given by

$$\sin x = A, \text{ and } \cos x = \sqrt{1-A^2}.$$

Compute $\cos 6x$ in terms of A.

(5) Use complex exponentials to calculate.

$$\int e^{3x}\sin(bx)dx, \qquad b \in \mathbb{R}$$

(6) Solve the following integral using complex numbers

$$\int \sin^2 3x \cos 5x dx$$

(7) Find all complex roots of the equation

$$z^3 + 3 = 0$$

(8) Find

$$\mathsf{Re}\left\{\frac{e^{(1-i)x}}{1+2i}\right\}$$

(9) (a) Find the general solution of $\frac{dy}{dx} = (\cos y)^2(1+x)$.
(b) Which solution satisfies $y(0) = \frac{\pi}{3}$?

(10) (a) Find the solution of $\begin{cases} x\frac{dy}{dx} + 3y = 2 + x^2 \\ y(1) = B \end{cases}$
(b) For which value of B does the limit $\lim_{x\to 0} y(x)$ exist?

(11) (a) Find the general solution of $5\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 5y = 0.$

(b) For which solution(s) $y(x)$ of the equation in part (i), does the limit $\lim_{x\to\infty} y(x)$ exist?

(12) (a) Find the solution of

$$\frac{d^2y}{dx^2} - 4y = 0 \quad y(0) = A, \quad y'(0) = -4.$$

(A is an unspecified constant.)

(b) How should you choose A if you want the solution to satisfy $\lim_{x\to\infty} y(x) = 0$?

(13) Find a particular solution of the equation

$$\frac{dy^2}{dx^2} - 2\frac{dy}{dx} - 3y = x + \sin x + e^{3x}.$$

(14) Find a particular solution of the equation

$$\frac{d^3y}{dx^3} - 4\frac{dy^2}{dx^2} + 2\frac{dy}{dx} + 3y = x + \sin x.$$

(15) Find the general solution of the equation

$$\frac{dy}{dx} + \frac{1+x}{1+y} = 0.$$

(16) Solve the following initial value problem

$$\frac{dy}{dx} + (\sin x)y = e^{\cos x}, \qquad y(\pi) = 0.$$

(17) (a) Find the general solution of $\quad 2\frac{d^2y}{dx^2} - \frac{dy}{dx} - 3y = 0.$

(b) Which solution satisfies $y(0) = 1$ and $y'(0) = 0$?

(18) Find the real form of the general solution of $y^{(4)} - 4y'' - 5y = 0$.

(19) Assume an electric circuit has a linear input voltage $V_{in} = At + B$, a capacitor $C = 1$, an inductance $L = 1$ and a resistor $R > 0$. From class we know that the current $I(t)$ satisfies the differential equation

$$\frac{d^2I}{dt^2} + R\frac{dI}{dt} + I = A.$$

(a) Find the current $I(t)$ assuming $A > 0$. Your answer will depend on R, and you'll find that you get different formulas for $R > 2$, $R = 2$ and $0 < R < 2$,

(b) Analyze what happens to the current as time grows (i.e. as $t \to \infty$) in the three cases $R > 2$, $R = 2$ and $0 < R < 2$.

(20) Find a particular solution of the equation

$$\frac{d^2y}{dt^2} + \frac{dy}{dt} = 6\cos 2t + 2.$$

Some answers to the problems

Don't read these answers before you have worked on the problems long enough!

1a In these type of problems you need to write the formula for the error $R_n f(x)$ and show that it goes to zero as $n \to \infty$, for any x. The formula is

$$R_n f(x) = \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1}$$

where $0 < c < x$ or $x < c < 0$, depending on the sign of x, and where *c changes with n*. In the case $f(x) = \cos(3x - 2)$, the error is

$$R_n f(x) = \frac{\pm\cos c}{(n+1)!}(3x-2)^{n+1}$$

or

$$R_n f(x) = \frac{\pm \sin c}{(n+1)!}(3x-2)^{n+1}$$

depending on n, where $0 < c < 3x-2$ or $3x-2 < c < 0$, depending on x. In both cases

$$|R_n f(x)| \le \frac{1}{(n+1)!}(3x-2)^{n+1}.$$

As $n \to \infty$, $\frac{A^n}{n!} \to \infty$, for any A. If we call $A = 3x-2$, we have it. The main point is that A and its coefficient should not depend on n, so be sure c is out of the way before taking that limit.

In the second case you have a similar situation but $e^c < e^{x^2}$, rather than 1. This does not depend on n, so it is not a problem once we take the limit.

1b Calculating the derivatives of $g(x) = e^{x^2}$ quickly gets complicated. Instead use the Taylor expansion with remainder for $h(t) = e^t$, and substitute $t = x^2$. You get

$$e^t = 1 + t + \cdots + \frac{t^n}{n!} + \underbrace{e^c \frac{t^{n+1}}{(n+1)!}}_{\text{remainder}}$$

where $0 < c < t$ or $t < c < 0$ (depending on the sign of t). Now set $t = x^2$ and you get

$$e^{x^2} = \underbrace{1 + x^2 + \cdots + \frac{x^{2n}}{n!}}_{\text{Taylor polynomial of degree } 2n} + \underbrace{e^c \frac{(x^2)^{n+1}}{(n+1)!}}_{\text{remainder}}$$

where $0 < c < x^2$. Therefore the remainder satisfies

$$|\text{Remainder}| \le e^{x^2} \frac{(x^2)^{n+1}}{(n+1)!}.$$

As $n \to \infty$ the right hand side here goes to zero because e^{x^2} doesn't depend on n, and $\frac{(x^2)^{n+1}}{(n+1)!} = \frac{a^{n+1}}{(n+1)!}$ with $a = x^2$. By the Sandwich theorem the Remainder also goes to zero as $n \to \infty$.

2a Two approaches: (1) imitate figure 11 from the notes with $a = 1$, $b = 2$. (2) read from the figure that $z = \frac{3}{2} - \frac{i}{2}$, multiply with $1 + 2i$ and draw the result.

2b If you did part (a) carefully you would have noticed that $(1+2i)z$ is exactly $2w$. Therefore $2w/z = 1 + 2i$ and $w/z = \frac{1}{2} + i$.

3a A complex number is negative and real if its argument is π (up to a multiple of 2π).
Make a drawing and you find that $\arg z = \pi/3$. Therefore $\arg z^2 = 2\pi/3$ and $\arg z^3 = \pi$. It follows that z^3 is the first power of z which is a negative real number.

3b A complex number is positive and real if its argument is a multiple of 2π.
Since $\arg z = \pi/3$ the first power of z which is positive real is z^6.

3c $\arg\left(-\sqrt{3} + i\right) = \frac{5}{6}\pi$, so $\left(-\sqrt{3} + i\right)^6$ is negative real, and $\left(-\sqrt{3} + i\right)^{12}$ is positive real.

3d $\arg\left(-2 - 2i\right) = \frac{5}{4}\pi$, so $\left(-2 - 2i\right)^4$ is negative real and $\left(-2 - 2i\right)^8$ is positive real.

3e If $z^2 = -i$ then $|z|^2 = |-i| = 1$ so $|z| = 1$.
Moreover, $2 \arg z = \arg z^2 = \arg(-i) = \frac{3}{2}\pi + 2k\pi$ (k an integer); in other words, $2 \arg z$ is $\frac{3}{2}\pi$ plus a multiple of 2π.
Divide by 2, and you find that $\arg z = \frac{3}{4}\pi + k\pi$, i.e. $\arg z$ is $\frac{3}{4}\pi$ plus a multiple of π.
This leads to two possibilities for z:

$$|z| = 1, \arg z = \tfrac{3}{4}\pi \implies z = -\tfrac{1}{2}\sqrt{2} + \tfrac{i}{2}\sqrt{2},$$

and

$$|z| = 1, \arg z = \tfrac{3}{4}\pi + \pi \implies z = \tfrac{1}{2}\sqrt{2} - \tfrac{i}{2}\sqrt{2},$$

3f $|z|^3 = |z^3| = |i| = 1 \implies |z| = 1.$
$3\arg z = \arg z^3 = \arg i = \frac{\pi}{2} + 2k\pi$ (k an integer). Hence $\arg z = \frac{\pi}{6} + k\frac{2\pi}{3}$, i.e. $\arg z$ is $\frac{\pi}{6}$ plus a multiple of $\frac{2\pi}{3}$ (120°).
This leads to three possibilities

$$\begin{aligned} |z| = 1, \arg z = \tfrac{\pi}{6} \qquad & \implies z = \tfrac{1}{2}\sqrt{3} + \tfrac{1}{2}i \\ |z| = 1, \arg z = \tfrac{1}{6}\pi + \tfrac{2}{3}\pi = \tfrac{5}{6}\pi \qquad & \implies z = -\tfrac{1}{2}\sqrt{3} + \tfrac{1}{2}i \\ |z| = 1, \arg z = \tfrac{1}{6}\pi + \tfrac{4}{3}\pi = \tfrac{3}{2}\pi \qquad & \implies z = -i. \end{aligned}$$

3g $|z|^2 = |z|^2 = |1+i| = \sqrt{2} \implies |z| = 2^{1/4}$
$2\arg z = \arg z^2 = \arg(1+i) = \frac{1}{4}\pi + 2k\pi$, so $\arg z = \frac{1}{8}\pi + k\pi$. This leads to two solutions

$$\begin{aligned} |z| = 2^{1/4}, \arg z = \tfrac{1}{8}\pi \qquad & \implies z = 2^{1/4}\cos\tfrac{\pi}{8} + i2^{1/4}\sin\tfrac{\pi}{8} \\ |z| = 2^{1/4}, \arg z = \tfrac{1}{8}\pi + \pi \qquad & \implies z = -2^{1/4}\cos\tfrac{\pi}{8} - i2^{1/4}\sin\tfrac{\pi}{8} \end{aligned}$$

Since $\pi/8$ is not one of the "familiar angles" we don't simplify $\sin\pi/8$ and $\cos\pi/8$ (even though you could do this using the half-angle formulas).

4 You need to write $\cos 6x$ in terms of $\cos x$ and $\sin x$. Since

$$e^{6xi} = \cos 6x + i\sin 6x$$

on one hand, and

$$e^{6ix} = (e^{ix})^6 = (\cos x + i\sin x)^6$$

on the other, we make those two equal. From the Pascal triangle we have the binomial expansion

$$(a+b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$$

and so

$$\begin{aligned} \cos 6x + i\sin 6x = \cos^6 x + 6i\cos^5 x\sin x + 15i^2\cos^4 x\sin^2 x + 20i^3\cos^3 x\sin^3 x \\ +15i^4\cos^2 x\sin^4 x + 6i^5\cos x\sin^5 x + i^6 b\sin^6 x. \end{aligned}$$

Equating the real parts in both sides

$$\cos 6x = \cos^6 x - 15\cos^4 x\sin^2 x + 15\cos^2 x\sin^4 x - \sin^6 x.$$

Substitute the given values $\sin^2 x = A$, and $\cos^2 x = 1 - A^2$ and you're done:

$$\cos 6x = (1-A^2)^3 - 15(1-A^2)^2A^2 + 15(1-A^2)A^4 - A^6.$$

(you don't have to simplify this any further).

5

$$\begin{aligned} \int e^{3x}\sin(bx)dx = \mathrm{Im}\int e^{3x}e^{ibx}dx = \mathrm{Im}\left[\frac{e^{(3+ib)x}}{3+ib}\right] + C = \mathrm{Im}\left[\frac{e^{(3+ib)x}}{9+b^2}(3-ib)\right] + C \\ = \frac{3}{9+b^2}e^{3x}\sin(bx) - \frac{b}{9+b^2}e^{3x}\cos(bx) + C \end{aligned}$$

6

$$\begin{aligned} \int \sin^2 3x\cos 5x dx = -\frac{1}{8}\int (e^{3xi} - e^{-3xi})^2(e^{5xi} + e^{-5xi})dx \\ = -\frac{1}{8}\int e^{11ix} + e^{-11ix} + e^{ix} + e^{-ix} - 2(e^{5ix} + e^{-5ix})dx \\ = -\frac{1}{4}\int \cos 11x + \cos x - 2\cos 5x dx = -\frac{1}{4}\left(\frac{1}{11}\sin 11x + \sin x - \frac{2}{5}\sin 5x\right) + C. \end{aligned}$$

7 We need $z^3 = -1 = e^{\pi i}$. If $z = re^{i\theta}$, then $z^3 = r^3e^{3i\theta} = e^{\pi i}$. From here $r^3 = 1$ so that $r = 1$, and $3\theta = \pi, 3\pi, 5\pi$ which gives $\theta = \frac{\pi}{3}, \pi, \frac{5}{3}\pi$. The three solutions are

$$z = e^{\frac{\pi}{3}i} = \frac{1}{2} + \frac{\sqrt{3}}{2}i, \quad z = e^{\pi i} = -1, \quad z = e^{\frac{5}{3}i\pi} = \frac{1}{2} - \frac{\sqrt{3}}{2}i.$$

8

$$\frac{e^{(1-i)x}}{1+2i} = \frac{e^{(1-i)x}}{1+4}(1-2i) = \frac{e^x}{5}(\cos x - i\sin x)(1-2i)$$

Therefore

$$\mathrm{Re}\left\{\frac{e^{(1-i)x}}{1+2i}\right\} = \frac{e^x}{5}\cos x - 2\frac{e^x}{5}\sin x.$$

9a This is a separable equation. Write it as

$$\frac{y'}{(\cos y)^2} = 1 + x,$$

and integrate

$$\tan y = \int \frac{dy}{\cos^2 y} = \int (1+x)dx = x + \frac{x^2}{2} + C.$$

9b The function is given implicitly and if we want $y(0) = \frac{\pi}{3}$, then $\tan\frac{\pi}{3} = \sqrt{3} = C$.

10a Write the equation as

$$y' + \frac{3}{x}y = \frac{2}{x} + x^2$$

so that the integrating factor is $e^{\int \frac{3}{x}dx} = e^{3\ln x} = x^3$. If we multiply by the factor we have

$$x^3y' + 3x^2y = 2x^2 + x^5, \qquad (x^3y)' = 2x^2 + x^5$$

$$x^3y = \frac{2}{3}x^3 + \frac{1}{6}x^6 + C, \qquad y = \frac{2}{3} + \frac{1}{6}x^3 + Cx^{-3}.$$

If we want $y(1) = B$ we need $\frac{2}{3} + \frac{1}{6} + C = B$ or $C = B - \frac{5}{6}$.

10b If we want $\lim_{x\to 0} y(x)$ to exist, we need $C = 0$ or $B = \frac{5}{6}$.

11a The characteristic equation for this problem is

$$5r^2 + 6r + 5 = 0$$

which is solved for the values $r = -\frac{3}{5} \pm \frac{4}{5}i$. Therefore, the general solution is

$$y = C_1e^{-\frac{3}{5}x}\cos\tfrac{4}{5}x + C_2e^{-\frac{3}{5}x}\sin\tfrac{4}{5}x.$$

11b Both $e^{-\frac{3}{5}x}\cos\frac{4}{5}x$ and $e^{-\frac{3}{5}x}\sin\frac{4}{5}x$ go to 0 as $x \to \infty$. Therefore y always goes to zero as $x \to +\infty$.

12a The characteristic equation is

$$r^2 - 4 = 0$$

which solves for $r = \pm 2$. The general solution is

$$y = C_1e^{2x} + C_2e^{-2x}.$$

If we impose the condition $y(0) = A, y'(0) = -4$, we get $C_1 + C_2 = A$, $2(C_1 - C_2) = -4$ or $C_1 = \frac{A}{2} - 1$, $C_2 = \frac{A}{2} + 1$.

12b The constant C_1 must be zero. If it isn't then the limit does not exist, while $C_1 = 0$ implies $\lim_{x\to\infty} y(x) = \lim_{x\to\infty} C_2e^{-2x} = 0$. So we must have $C_1 = \frac{A}{2} - 1 = 0$, whence $A = 2$.

13 First we must solve the characteristic equation

$$r^2 - 2r - 3 = 0$$

which gives $(r+1)(r-3) = 0$. The general solution to the homogeneous is

$$y_h = C_1e^{-x} + C_2e^{3x}.$$

We now try a particular solution for each $x, \sin x$ and e^{3x}. For the first one we try $y_1 = a + bx$, for the second one $A\sin x$ and for the third Bxe^{3x} (notice that e^{3x} is a solution of the homogeneous, so we need to multiply by x. Substitute each and solve for the constants.

14 We are not asked to find the general solution, only one particular solution. Looking at the right hand side we decide to try a particular solution which combines $y_p(x) = Ax + B$ (to get the "x" term) and $y_p(x) = C\cos x + D\sin x$ (to get the "$\sin x$" term). So we try

$$y_p(x) = Ax + B + C\cos x + D\sin x$$

Substitution gives

$$y_p''' - 4y_p'' + 2y_p' + 3y_p = 3Ax + (3B + 2A) + (7C + D)\cos x + (7D - C)\sin x,$$

so we want the coefficients A, B, C, D to satisfy

$$3A = 1, \quad 3B + 2A = 0, \quad 7C + D = 0, \quad 7D - C = 1.$$

Solving these equations leads to

$$A = \tfrac{1}{3}, \quad B = -\tfrac{2}{9}, \quad C = -\tfrac{1}{50}, \quad D = \tfrac{7}{50}.$$

15 A separable equation

$$y'(1+y) = -(1+x), \qquad \int (1+y)dy = -\int (1+x)dx$$

$$2y + y^2 = -2x - x^2 + C$$

for an arbitrary constant C, which gives y implicitly. One can also solve for y

$$(y+1)^2 - 1 = -(x+1)^2 + 1 + C,$$

or, since C is arbitrary

$$(y+1)^2 = C - (x+1)^2, \qquad y = \pm\sqrt{C - (x+1)^2} - 1$$

the sign will depend on the value of the initial condition $y(0)$.

16 One chooses an integrating factor here, namely

$$e^{\int \sin x dx} = e^{-\cos x}$$

so that, after multiplying the equation becomes

$$(ye^{-\cos x})' = 1$$

or

$$y = e^{\cos x}(x + C).$$

If $y(\pi) = 0$, then $e^{-1}(\pi + C) = 0$ or $C = -\pi$.

17 The characteristic roots are -1, $\frac{3}{2}$, general solution is $y(x) = Ae^{-x} + Be^{3x/2}$, The solution with given initial values has $A = \frac{3}{5}$, $B = \frac{2}{5}$, so $y(x) = \frac{3}{5}e^{-x} + \frac{2}{5}e^{3x/2}$.

18 Try $y = e^{rt}$. The characteristic equation is $r^4 - 4r^2 - 5 = 0$, which you can write as

$$\left(r^2\right)^2 - 4r^2 - 5 = 0 \iff \left(r^2 - 5\right)\left(r^2 + 1\right) = 0$$

so the roots are

$$r_{1,2} = \pm\sqrt{5}, \quad r_{3,4} = \pm i.$$

The general solution is

$$y(t) = Ae^{-\sqrt{5}t} + Be^{\sqrt{5}t} + C\cos t + D\sin t.$$

19a We first solve for the homogeneous equation. The characteristic polynomial is

$$r^2 + Rr + 1 = 0$$

and so $r = -\frac{R}{2} \pm \frac{1}{2}\sqrt{R^2 - 4}$. Call these two roots r_1 and r_2. The roots are real if $R \geq 2$, they coincide when $R = 2$, and they are complex for $0 < R < 2$.

If $R > 2$, then the general solution of the homogeneous equation is

$$y_h = C_1 e^{r_1 t} + C_2 e^{r_2 t}.$$

Choose as particular solution $y_p = C$ for some constant C, and we find $C = A$ after substituting. Hence the general solution is

$$y = A + C_1 e^{r_1 t} + C_2 e^{r_2 t}.$$

If $R = 2$, then $r = -\frac{1}{2}R = -1$ is a double root. The solution of the homogeneous equation is

$$y_h = C_1 e^{-t} + C_2 t e^{-t}$$

and a particular solution is again $y_p = A$. The general solution is

$$y = A + C_1 e^{-t} + C_2 t e^{-t}.$$

If $0 < R < 2$, then we have two complex roots and the general solution of the homogeneous equation is

$$y_h = C_1 e^{-\frac{1}{2}Rt} \cos(\frac{t}{2}\sqrt{4 - R^2}) + C_2 e^{-\frac{1}{2}Rt} \sin(\frac{t}{2}\sqrt{4 - R^2}).$$

Again, the particular solution is $y_p = A$ and so the general solution is

$$y = A + C_1 e^{-\frac{1}{2}Rt} \cos(\frac{t}{2}\sqrt{4 - R^2}) + C_2 e^{-\frac{1}{2}Rt} \sin(\frac{t}{2}\sqrt{4 - R^2}).$$

19b In the case $R > 2$, both r_1 and r_2 are negative, thus $\lim_{t\to\infty} y(t) = A$. If $R = 2$, the behavior is the same and $\lim_{t\to\infty} y(t) = A$. Finally, if $0 < R < 2$, cosine and sine alternate between 1 and -1. but the exponential part goes to zero. Again $\lim_{t\to\infty} y(t) = A$.

20 The solution of the homogeneous is found through the characteristic equation

$$r^2 + r = 0$$

with solutions $r = 0, -1$. Thus

$$y_h = C_1 + C_2 e^{-t}.$$

A choice of particular solution is $y_p = at + b\cos 2t + c\sin 2t$. Substituting we find $a = 2$ and $2c - 4b = 6$ and $-4c - 2b = 0$ or $c = \frac{3}{5}$, $b = -\frac{6}{5}$.