# MATH 222 — THE FIRST MIDTERM — Lecture 2

February 24, 2009 $7^{15}$– $8^{45}$pm

**1.** Evaluate the following integrals.

(a) Compute $\int \frac{\sin x}{4+\cos^2 x}dx$ and also $\int \frac{\tan x}{4\sec x+\cos x}dx$ .

**Solution:** Substitute $u = \cos x$ to get

$$\int \frac{\sin x}{4+\cos^2 x}dx = -\int \frac{du}{4+u^2}.$$

Then substitute $u = 2v$ to get

$$-\int \frac{du}{4+u^2} = -\frac{2}{4}\int \frac{dv}{1+v^2} = -\tfrac{1}{2}\arctan v + C.$$

Undo the substitutions:

$$\int \frac{\sin x}{4+\cos^2 x}dx = -\tfrac{1}{2}\arctan\frac{\cos x}{2} + C.$$

To do the second integral, first get rid of the nefarious secant:

$$\frac{\tan x}{4\sec x+\cos x} = \frac{\frac{\sin x}{\cos x}}{\frac{4}{\cos x}+\cos x} = \frac{\sin x}{4+\cos^2 x}.$$

So the second integral is the same as the first.

(b) $\int_1^2 x^2 e^{ax}dx$, where $a$ is any constant.

**Solution:** Integrate by parts twice.

$$\begin{aligned}\int_1^2 x^2e^{ax}dx &= \Big[\frac{e^{ax}}{a}x^2\Big]_1^2 - \frac{2}{a}\int_1^2 xe^{ax}dx\\ &= \Big[\frac{e^{ax}}{a}x^2 - \frac{2}{a^2}e^{ax}x\Big]_1^2 + \frac{2}{a^2}\int_1^2 e^{ax}dx\\ &= \Big[\frac{e^{ax}}{a}x^2 - \frac{2}{a^2}e^{ax}x + \frac{2}{a^3}e^{ax}\Big]_1^2\\ &= \Big(\frac{1}{a}2^2 - \frac{2}{a^2}2^1 + \frac{2}{a^3}\Big)e^{2a} - \Big(\frac{1}{a} - \frac{2}{a^2} + \frac{2}{a^3}\Big)e^a\end{aligned}$$

**2.** (a) Find the reduction formula for the integral

$$I_n = \int x^3 (\ln x)^n\,dx.$$

**Solution:** Integrate by parts, integrating the $x^3$

$$\begin{aligned}I_n &= \int x^3(\ln x)^n dx\\ &= \frac{x^4}{4}(\ln x)^n - \frac{n}{4}\int x^4(\ln x)^{n-1}\frac{1}{x}dx\\ &= \frac{x^4}{4}(\ln x)^n - \frac{n}{4}\int x^3(\ln x)^{n-1}dx\end{aligned}$$

so that

$$I_n = \frac{x^4}{4}(\ln x)^n - \frac{n}{4}I_{n-1}.$$

(b) Calculate the integral

$$\int x^3 (\ln x)^2 \, dx.$$

You do not need to simplify.

**Solution:** We are asked to find $I_2$. Applying the reduction formula twice will give us a formula for $I_2$ involving $I_0$, namely

$$\begin{aligned} I_2 &= \frac{x^4}{4}(\ln x)^2 - \frac{2}{4} I_1 \\ &= \frac{x^4}{4}(\ln x)^2 - \frac{2}{4}\left\{\frac{x^4}{4}(\ln x)^1 - \frac{1}{4} I_0\right\}. \end{aligned}$$

Since $(\ln x)^0 = 1$, the last integral is

$$I_0 = \int x^3 dx = \frac{x^4}{4} + C,$$

so we get

$$I_2 = \frac{x^4}{4}(\ln x)^2 - \frac{2}{4}\left\{\frac{x^4}{4}(\ln x)^1 - \frac{1}{4}\frac{x^4}{4}\right\} + C. \qquad \text{(not the same } C \text{ as above)}$$

We are not asked to simplify, but nonetheless, this is the same as

$$I_2 = \frac{x^4}{4}\left\{(\ln x)^2 - \tfrac{1}{2}\ln x + \tfrac{1}{8}\right\} + C.$$

**3.** (a) Calculate $\int \frac{2dx}{(x+1)(x^2+1)}$. Explain the steps in the method you plan to use.

**Solution:** We find the Partial Fraction Expansion of the integrand.

$$\frac{2}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}.$$

The Heaviside trick will give us $A$ but not $B$ or $C$. So we multiply both sides with $(x+1)(x^2+1)$ and expand:

$$2 = A(x^2+1) + (Bx+C)(x+1) = (A+B)x^2 + (B+C)x + (A+C)$$

which gives us the following equations for $A, B, C$

$$A + B = 0, \quad B + C = 0, \quad A + C = 2.$$

The solution is $A = C = 1$, $B = -1$, so

$$\frac{2}{(x+1)(x^2+1)} = \frac{1}{x+1} + \frac{-x+1}{x^2+1} = \frac{1}{x+1} - \frac{x}{x^2+1} + \frac{x}{x^2+1}.$$

and

$$\int \frac{2dx}{(x+1)(x^2+1)} = \ln|x+1| - \tfrac{1}{2}\ln(x^2+1) + \arctan x + C.$$

(b) Find the Partial Fraction Expansion of the function $f(x) = \frac{3x^2+9}{(x^2-1)^2(x^2+2x+2)^2}$. You do not have to find the coefficients "$A, B, \ldots$".

**Solution:** The degree of the numerator is two, which is less than the degree of the denominator (eight).
Factor the denominator:
$x^2 + 2x + 2 = (x+1)^2 + 1$ so $x^2 + 2x + 2$ can't be factored any further.
$x^2 - 1 = (x-1)(x+1)$.

Therefore we get

$$\frac{3x^2+9}{\left(x^2-1\right)^2\left(x^2+2x+2\right)^2} = \frac{3x^2+9}{(x+1)^2(x-1)^2\left(x^2+2x+2\right)^2}$$
$$= \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1} + \frac{D}{(x+1)^2} +$$
$$\frac{Kx+L}{x^2+2x+2} + \frac{Mx+N}{\left(x^2+2x+2\right)^2}.$$

**4.** (a) Find the Taylor-Maclaurin expansions up to $o(x^4)$ of

$$f(x) = 2\arctan x - \sin 2x \text{ and } g(x) = e^{2x} - 1.$$

**Solution:** We have the series for $e^x$ and $\arctan x$ memorized (and if we forget the one for $\arctan x$, then we can find it again by integrating the series for $1/(1+x^2)$.)
We have

$$\begin{aligned}
2\arctan x &= 2x - 2\frac{x^3}{3} + o(x^4) && \text{(from the series for arctan)}\\
\sin(2x) &= 2x - \frac{(2x)^3}{3!} + o(x^4) && \text{(from the series for the Sine)}\\
&= 2x - \frac{4}{3}x^3 + o(x^4)\\
\Longrightarrow f(x) &= \frac{2}{3}x^3 + o(x^4) && \text{(subtract)}
\end{aligned}$$

For $g(x)$ we have

$$\begin{aligned}
g(x) &= e^{2x} - 1\\
&= \left[1 + (2x) + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \frac{(2x)^4}{4!} + o(x^4)\right] - 1\\
&= 2x + 2x^2 + \tfrac{4}{3}x^3 + \tfrac{2}{3}x^4 + o(x^4)
\end{aligned}$$

(b) Compute $\lim_{x\to 0} \frac{2\arctan x - \sin 2x}{x^2(e^{2x}-1)}$

**Solution:** Using the series for $f(x)$ and $g(x)$ from part (a) we find that

$$\begin{aligned}
\frac{2\arctan x - \sin 2x}{x^2(e^{2x}-1)} &= \frac{f(x)}{x^2 g(x)} && \text{(use part (a))}\\
&= \frac{\frac{2}{3}x^3 + o(x^4)}{x^2\left(2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4 + o(x^4)\right)}\\
&= \frac{\frac{2}{3}x^3 + o(x^4)}{2x^3 + 2x^4 + o(x^4)} && \text{(divide top\&bottom by } x^3\text{)}\\
&= \frac{\frac{2}{3} + \frac{o(x^4)}{x^3}}{2 + 2x + \frac{o(x^4)}{x^3}} && \left(\text{use } \frac{o(x^4)}{x^3} = o(x)\right)\\
&= \frac{\frac{2}{3} + o(x)}{2 + 2x + o(x)}
\end{aligned}$$

Therefore

$$\lim_{x\to 0} \frac{2\arctan x - \sin 2x}{x^2(e^{2x}-1)} = \lim_{x\to 0} \frac{\frac{2}{3} + o(x)}{2 + 2x + o(x)} = \frac{2/3}{2} = \frac{1}{3}.$$

**5.** Use Lagrange's formula for the remainder term in the Taylor series of $f(x) = \sin x$ to estimate the error we make when approximating the value of $\sin 1$ by $1 - \frac{1}{3!} = \frac{5}{6}$.

**Solution:** The Taylor-Maclaurin series of $\sin x$ is

$$T_\infty[\sin x] = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots.$$

If we set $x = 1$ in this series and only add the first two terms above we get $1 - 1/3! = 5/6$. The first two terms add up to $x - x^3/3!$ which is the Taylor polynomial of degree **three** of the function $f(x) = \sin x$. So the question is *How much is the difference between* $f(1)$ *and* $T_3[f(1)]$ for the function $f(x) = \sin x$? The difference is, by definition, the remainder term

$$\text{error} = R_3 f(1) = f(1) - T_3[f(1)].$$

Lagrange's formula says that

$$R_3 f(1) = \frac{f^{(4)}(c)}{4!}(1)^4$$

for some $c$ between 0 and 1. The fourth derivative of $f(x) = \sin x$ is again $\sin x$, so

$$R_3 f(1) = \frac{\sin c}{24} \text{ for some } c \text{ with } 0 < c < 1.$$

We don't know anything else about $c$, so the only things we can say for sure about $\sin c$ are:

- Since $1 < \frac{\pi}{2}$ and since $\sin x$ is increasing between 0 and $\pi/2$, $\sin c$ can't be more than $\sin 1$, or less than $\sin 0$. Therefore the error satisfies

$$0 < \text{error} < \frac{\sin(1)}{24}.$$

  (several students arrived at this answer without checking if $\sin x$ is increasing between $x = 0$ and $x = 1$.)
- No matter what $c$ is, one always has $-1 \le \sin c \le +1$ so

$$|\text{error}| \le \frac{1}{24}.$$

Both answers are correct: the first is more accurate, the second is simpler to derive.

Some students remembered that $x - x^3/3!$ is not only the third degree but also the fourth degree Taylor-Maclaurin polynomial of $f(x) = \sin x$. Therefore the difference between $\sin(1)$ and $1 - 1/3!$ is also $R_4[\sin(1)]$, so that according to Lagrange the error is

$$R_4[\sin(1)] = \frac{f^{(5)}(c)}{5!}(1)^5$$

for some $c$ between 0 and 1. Since $f^{(5)}(x) = \cos x$ and since $\cos c$ always lies between $-1$ and $+1$, one finds that

$$|\text{error}| \le \frac{1}{120}.$$

This solution is also correct.

**6.** Find the Taylor-Maclaurin expansion of the function $f(x) = \frac{2x}{(1+x^2)^2}$.

Hint: $f(x) = F'(x)$ where $F(x) = -1/(1+x^2)$.

**Solution:** Using the geometric series we find that the Taylor-Maclaurin expansion of $F(x)$ is

$$T_\infty[F(x)] = -1 + x^2 - x^4 + x^6 - x^8 + \cdots + (-1)^{n+1}x^{2n} + \cdots$$

Therefore the Taylor-Maclaurin expansion of $f(x) = F'(x)$ is

$$T_\infty[f(x)] = 2x - 4x^3 + 6x^5 - 8x^7 + \cdots + (-1)^{n+1}2nx^{2n-1} + \cdots$$