MATH 222 — THE SECOND MIDTERM — LECTURE 2

THE EXAM

(1) (16pts) Prove that the Taylor series of $f(x) = \sin(2x^2)$ converges to $\sin(2x^2)$ for all real numbers x.

(2) (16pts)
  (a) Find r, θ such that $1 + i\sqrt{3} = re^{i\theta}$.
  (b) Find the smallest integer $n > 10$ such that $(1 + i\sqrt{3})^n$ is a real number.

(3) (17pts) Compute the following two real integrals by computing one complex integral

$$A = 4\int (\cos 2x)^2 \cos 3x \, dx, \qquad B = 4\int (\cos 2x)^2 \sin 3x \, dx,$$

(4) (16pts) Find the general solution of the differential equation

$$\frac{dy}{dx} + \frac{x}{1+x^2} y = x.$$

(5) (18pts)
  (a) Find the general solution of the differential equation

$$\frac{d^2y}{dt^2} = 3\frac{dy}{dt} + 10y.$$

  (b) Find the solution with $y(0) = 1$ and $y'(0) = A$. (The constant A may appear in your answer.)
  (c) How should you choose A if you want $\lim_{t\to\infty} y(t) = 0$?

(6) (17pts) A time changing voltage $V_{in}(t) = -2e^{-t} + \frac{3}{2}\sin 2t$ is applied to a resistor of 2 Ohms, an inductance of 1 Henry and a capacitor of $\frac{1}{5}$ Farads in an electric circuit. The differential equation modeling the current $I(t)$ of the circuit is given

$$\frac{d^2I}{dt^2} + 2\frac{dI}{dt} + 5I = 2e^{-t} + 3\cos 2t.$$

  (a) Find the general solution of this equation.
  (b) Describe the behavior of the general solution as $t \to +\infty$.

SOLUTIONS

1 You can get the series with remainder for $f(x) = \sin(2x^2)$ by substituting $t = 2x^2$ in the Taylor series with remainder for $g(t) = \sin t$. One has

$$g(t) = \underbrace{T_n(t)}_{\text{the Taylor poly}} + \underbrace{R_n(t)}_{\text{remainder}}$$

where according to Lagrange

$$R_n(t) = g^{(n+1)}(c)\frac{t^{n+1}}{(n+1)!} = \pm\{\sin(c) \text{ or } \cos(c)\}\frac{t^{n+1}}{(n+1)!}.$$

for some c between 0 and t. Since $|\sin c| \le 1$ and $|\cos c| \le 1$ we have

$$|R_n(t)| \le \frac{|t|^{n+1}}{(n+1)!}.$$

When you substitute $t = 2x^2$ you get

$$\sin(2x^2) = \underbrace{T_n(2x^2)}_{\text{the Taylor poly}} + \underbrace{R_n(2x^2)}_{\text{remainder}}.$$

So the remainder satisfies

$$|R_n(2x^2)| \le \frac{|2x^2|^{n+1}}{(n+1)!}.$$

The right hand side goes to zero as $n \to \infty$ and therefore the remainder also goes to zero.

2a $1 + i\sqrt{3} = 2e^{i\pi/3}$.

2b $(1+i\sqrt{3})^n = 2^n e^{n\pi/3\, i}$. This is real if $\frac{n\pi}{3}$ is a multiple of π so n must be a multiple of 3. The smallest multiple of 3 above 10 is $n = 12$.

3 Compute

$$A + iB = 4\int (\cos 2x)^2 e^{3ix}\, dx = \int \left(e^{2ix} + e^{-2ix}\right)^2 e^{3ix}\, dx = \cdots$$

You get

$$A = \tfrac{1}{7}\sin 7x + \tfrac{2}{3}\sin 3x + \sin x + C_1, \qquad B = -\tfrac{1}{7}\cos 7x - \tfrac{2}{3}\cos 3x + \cos x + C_2.$$

4 The equation is first order linear, so look for an integrating factor $m(x)$. Multiply the equation with $m(x)$:

$$my' + m\frac{x}{1+x^2}y = mx$$

and you see that you want $m' = mx/(1+x^2)$. Solving this for m leads to

$$\ln m = \int \frac{x\,dx}{1+x^2} = \tfrac{1}{2}\ln(1+x^2) \quad (\text{use } u = 1+x^2 \text{ to do the integral}).$$

Thus the integrating factor is $m(x) = \sqrt{1+x^2}$, from which you get

$$\frac{dy\sqrt{1+x^2}}{dx} = x\sqrt{1+x^2} \implies y\sqrt{1+x^2} = \int x\sqrt{1+x^2}\,dx = \frac{1}{3}\left(1+x^2\right)^{3/2} + C$$

whence

$$y = \tfrac{1}{3}(1+x^2) + \frac{C}{\sqrt{1+x^2}}$$

5b Characteristic equation is $r^2 - 3r - 10 = 0$, which has two real roots, $r_1 = 5$ and $r_2 = -2$, so the general solution is $y = C_1e^{5t} + C_2e^{-2t}$.

To achieve the specified initial values, we must have

$$y(0) = C_1 + C_2 = 1, \qquad y'(0) = 5C_1 - 2C_2 = A.$$

The solution of these equations is $C_1 = (A+2)/7$, $C_2 = (5-A)/7$, so the solution of the diffeq is

$$y(t) = \frac{A+2}{7}e^{5t} + \frac{5-A}{7}e^{-2t}.$$

5c For $\lim_{t\to\infty} y(t)$ to exist the coefficient in front of e^{5t} must be zero. So we must have $A = -2$.

6a The general solution $I(t)$ is the sum of the general solution to the homogeneous equation ($I_h(t)$) and any particular solution ($I_p(t)$) you find. To solve the homogeneous equation

$$\frac{d^2I}{dt^2} + 2\frac{dI}{dt} + 5I = 0$$

you try $I = e^{rt}$, which leads to the characteristic equation $r^2 + 2r + 5 = 0$. This factors as follows

$$r^2 + 2r + 5 = (r+1)^2 + 4 = (r+1+2i)(r+1-2i)$$

so that the characteristic roots are $r_{1,2} = -1 \pm 2i$. The general solution to the homogeneous equation is

$$I_h(t) = \underbrace{C_1 e^{(-1+2i)t} + C_2 e^{(-1-2i)t}}_{\text{complex form}} = \underbrace{Ae^{-t}\cos 2t + Be^{-t}\sin 2t.}_{\text{real form}}$$

To find a particular solution try $I_p(t) = Ke^{-t} + L\cos 2t + M\sin 2t$. Then

$$I_p'' + 2I_p' + 5I_p = 4Ke^{-t} + (L + 4M)\cos 2t + (M - 4L)\sin 2t.$$

We want the right hand side to be equal to $2e^{-t} + 3\cos 2t$, so we get these equations for K, L, M

$$4K = 2, \quad L + 4M = 3, \quad M - 4L = 0.$$

Solving these equations gives us $K = \frac{1}{2}$, $L = \frac{3}{17}$, $M = \frac{12}{17}$. The general solution is therefore

$$I(t) = I_h(t) + I_p(t) = Ae^{-t}\cos 2t + Be^{-t}\sin 2t + \tfrac{1}{2}e^{-t} + \tfrac{3}{17}\cos 2t + \tfrac{12}{17}\sin 2t.$$

6b As $t \to \infty$ the terms containing e^{-t} go to zero, so the only remaining terms are

$$I(t) \approx \tfrac{3}{17}\cos 2t + \tfrac{12}{17}\sin 2t.$$

For large t the solution oscillates.