STUDY GUIDE AND SAMPLE PROBLEMS FOR THE FINAL EXAM
MATH 222, SPRING 2009

Notice that the Final is cumulative. Below is a list of topics from the last part of the course, for material on previous midterms, please see the other study guides.

List of topics for the Final exam

- Vectors: Learn the arithmetics of vectors, how to add them, multiply them by numbers, etc. Be careful to always know what you are dealing with so you don't add a 2-vector to a 3-vector or a number to a vector. Learn the difference between a point, a position vector, a general vector and how the coordinates of all of these are related (this is in 40.3)
- Parametric equations of lines and planes: Understand what they are and how to get them.
- Dot product: Learn the definition, its properties and how to apply to simplify and rewrite expressions involving lengths. Learn its relation to the cosine of the angles between two vectors.
- Orthogonal projection: try not to memorize, instead write $\vec{\boldsymbol{x}} = \lambda\vec{\boldsymbol{a}} + \vec{\boldsymbol{y}}$ and take the dot product with $\vec{\boldsymbol{a}}$ to find λ. From here $\lambda\|\vec{\boldsymbol{a}}\|$ will be the length of the projection. Understand what is going on. . .
- Defining equations of lines and planes: you write this equation in terms of one point and the normal vector. In the case of lines in the plane you should know how to relate this to the parametric equation. There are many multiple step problems that can appear here, like in example 43.8.
- Distance from a point to a line in two dimensions, or to a plane in three dimensions: learn how to find it.
- Cross product: definition, including the use of determinants. You need to know its length and its direction, its algebraic properties and its relation to the triple product. You also need to know how to apply it to calculate the area of parallelograms, the volume of parallelepipeds and the normal to a plane.

- Vector functions and parametrized curves: learn what they are and what they represent. Know the formula of the standard ones (circle, helix, cycloid, lines). Find derivatives and know their interpretations as velocity and acceleration vectors. Calculate speed and know the relation between the acceleration vector and the force one feels when in a movement. Learn how to differentiate dot and cross products.
- Formula of the tangent line in terms of the velocity: learn how to find the tangent line, with perhaps some follow-up questions (find intersections, etc.)
- Sketch parametrized curves: practice sketching curves, including those in polar coordinates (i.e. given $r = f(\theta)$ sketch $\vec{\boldsymbol{x}}(\theta) = \begin{pmatrix} f(\theta)\cos\theta \\ f(\theta)\sin\theta \end{pmatrix}$.)
- Length of a curve: learn to calculate the length of some simple curves. Some of these integrals might be a little tricky.

Practice problems from old exams

(1) (a) Consider the plane going through the points $A = (1, 2, 3)$, $B = (4, 5, 6)$, $C = (1, 1, 1)$.
   (i) Find two vectors on the plane.
   (ii) Find a normal vector to the plane.
   (iii) Find the equation of the plane.
   (iv) Find the points where the plane intercepts the x, y and z axis.
   (v) Draw the plane.

(b) Consider the straight line going through the point $D = (1, 0, 1)$ and having the direction of the vector $\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$

(i) Find the equation of the line.

(ii) Is the line orthogonal to the plane in part (a)? Explain.

(iii) Does the line intersect the plane in part (a)? If so, at which point?

(2) Given the points $A = (0, 1, 0)$, $B = (2, 0, 0)$, $C = (1, 0, -1)$, $D = (0, 1, -2)$

(a) Find the volume of the parallelepiped whose sides are the vectors AB, AC and AD.

(b) Find the distance from B to the plane ACD.

(c) Find the area of the parallelogram formed by AC and AD.

(d) Using (b) and (c), check that your result in (a) was correct.

(3) Given the parametric curve

$$\vec{\boldsymbol{x}}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 1 - t^2 \\ \sqrt{2}t^4 \end{pmatrix}$$

(a) Find the velocity and acceleration vectors.

(b) Find the intersections of the curve with the y-axis and find the angle it forms at the points of intersection (that is, the angle between the tangent vector and the y-axis).

(4) Consider the polar graph

$$r = 2(1 - \cos\theta)$$

(a) Write its Cartesian coordinates.

(b) Find the intersection with x and y axes.

(c) Find the values of θ for which the velocity vector is vertical.

(d) Sketch its graph.

(5) Consider the points $A(2, 2, 0)$, $B(1, 2, 1)$, and $C(1, 0, h)$ where h is a constant (which may appear in your answers below).

(a) Find the coordinates of the point D for which $ABDC$ is a parallelogram.

(b) Find the area of the triangle ABC.

(c) For which value of h is the triangle ABC a right triangle (with B being the right angle)?

(d) Find an equation for the plane through A, B, and C.

(e) For which value of h does the plane through ABC contain the origin?

(6) Consider the parametric curve $\vec{\boldsymbol{x}}(t) = \begin{pmatrix} t^2 \\ t^3 \end{pmatrix}$ with $0 \le t \le 2$.

(a) Where does the tangent line at the point with $t = 2$ intersect the x_1 and x_2 axes?

(b) Find the length of the curve segment with $-1 \le t \le 1$.

(7) Given are three points $A(1, 1, 3)$, $B(4, 0, 1)$ and $C(0, 1, 5)$, and a fourth point $D(1, h, h)$. Here h is an unknown constant. Let $\mathcal{P}$ be the plane through the points A, B, C.

(a) Find a vector perpendicular to the plane $\mathcal{P}$.

(b) Find the area of the triangle ABC.

(c) Find h so that D lies on the plane $\mathcal{P}$.

(8) Consider the parametric curve given by $\vec{\boldsymbol{x}}(t) = \begin{pmatrix} \cos t + t\sin t \\ \sin t - t\cos t \end{pmatrix}$.

(a) Let A be the point on the curve with $t = \pi/4$. Find a parametric representation for the tangent line to the curve at the point A.

(b) For which value of t with $0 \le t \le \pi$ is the tangent vector to our parametric curve perpendicular to the vector $\vec{\boldsymbol{v}} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$?

(c) Find the length of the curve segment corresponding to parameter values $0 \le t \le \pi/2$.

(9) Given are three points $A(0, 0, 2)$, $B(1, 2, 1)$ and $C(q, 0, 0)$ where q is an unknown constant.

(a) For which value(s) of q is the triangle ABC a right triangle with a right angle at C?

(b) Find an equation for the plane $\mathcal{P}$ through A, B, C.

(c) If the plane $\mathcal{P}$ contains the point $(0, 8, 0)$ then what is q?

(d) What is the distance from the origin to the plane $\mathcal{P}$ if $q = 4$?

(10) Consider the parametric curve defined by the vector function $\vec{\boldsymbol{x}}(t) = \begin{pmatrix} t^2 \\ t - t^3 \end{pmatrix}$.

(a) Find a parametric equation for the tangent to the curve at the point $\vec{\boldsymbol{x}}(t)$ when $t = 2$.

(b) For which values of t is the tangent line to the curve horizontal?

(c) Find an integral for the length of the segment of the curve $\vec{\boldsymbol{x}}(t)$, with $-1 \le t \le 1$. (You do not have to compute the integral.)

Some answers to the problems

SPOILER ALERT

Don't read these answers before you have worked on the problems long enough!

1(a)i $\overrightarrow{AB} = \begin{pmatrix} 3\\3\\3 \end{pmatrix}$, $\overrightarrow{AC} = \begin{pmatrix} 0\\-1\\-2 \end{pmatrix}$

1(a)ii $\vec{\boldsymbol{n}} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} -3\\6\\-3 \end{pmatrix}$, or you can also choose a multiple of $\vec{\boldsymbol{n}}$, for example $\begin{pmatrix} 1\\-2\\1 \end{pmatrix}$.

1(a)iii

$$0 = \vec{\boldsymbol{n}} \bullet \overrightarrow{AX} = (x-1) - 2(y-2) + z - 3 = x - 2y + z = 0.$$

1(a)iv All the interceptions are the same, $p = \begin{pmatrix} 0\\0\\0 \end{pmatrix}$.

1(b)i $\begin{pmatrix} x\\y\\z \end{pmatrix} = \begin{pmatrix} 1+2t\\t\\1 \end{pmatrix}$

1(b)ii No, the vector $\begin{pmatrix} 2\\1\\0 \end{pmatrix}$ is not parallel (a multiple of) the normal to the plane $\vec{\boldsymbol{n}}$.

1(b)iii No, if we substitute $\begin{pmatrix} x\\y\\z \end{pmatrix} = \begin{pmatrix} 1+2t\\t\\1 \end{pmatrix}$ in the equation of the plane $x - 2y + z = 0$ we get that $2 = 0$ which is not possible. Therefore, there is no intersection.

2a $\overrightarrow{AB} = \begin{pmatrix} 2\\-1\\0 \end{pmatrix}$, $\overrightarrow{AC} = \begin{pmatrix} 1\\-1\\-1 \end{pmatrix}$, $\overrightarrow{AD} = \begin{pmatrix} 0\\0\\-2 \end{pmatrix}$, $\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} 1\\2\\-1 \end{pmatrix}$.
The volume is $(\overrightarrow{AB} \times \overrightarrow{AC}) \bullet \overrightarrow{AD} = 2$.

2b A normal to the plane ACD is $\vec{\boldsymbol{n}} = \overrightarrow{AC} \times \overrightarrow{AD} = \begin{pmatrix} 2\\2\\0 \end{pmatrix}$. The distance from the plane to B is therefore $\frac{\vec{\boldsymbol{n}} \bullet (\vec{\boldsymbol{b}} - \vec{\boldsymbol{a}})}{\|\vec{\boldsymbol{n}}\|} = 1/\sqrt{2} = \frac{1}{2}\sqrt{2}$.

2c The area is $\|\overrightarrow{AC} \times \overrightarrow{AD}\| = 2\sqrt{2}$.

2d The volume of a parallelepiped is "Area base × height", which is $2\sqrt{2} \times \frac{1}{2}\sqrt{2}$ which is indeed 2.

3a $\vec{\boldsymbol{v}} = \begin{pmatrix} -2t\\4\sqrt{2}t^3 \end{pmatrix}$, $\vec{\boldsymbol{a}} = \begin{pmatrix} -2\\12\sqrt{2}t^2 \end{pmatrix}$.

3b This parametric curve traces out the part of the parabola $y = \sqrt{2}(1-x)^2$ with $x \le 1$ (solve $x = 1 - t^2$ for t^2 and substitute in $y = \sqrt{2}(t^2)^2$) The half-parabola gets traced out twice: once from left to right, and then back again.
$x(t) = 1 - t^2$ vanishes when $t = \pm 1$, so the points of intersection with the y-axis are $(0, 2)$. There is only one point of intersection.
At the point of intersection one has $t = \pm 1$, so the velocity vectors are $\vec{\boldsymbol{v}}(-1) = \begin{pmatrix} 2\\-4\sqrt{2} \end{pmatrix}$ and $\vec{\boldsymbol{v}}(1) = \begin{pmatrix} -2\\4\sqrt{2} \end{pmatrix}$.
To compute the cosine of the angle between the tangent and the y-axis you could use the dot product:

$$\cos\theta = \frac{\vec{\boldsymbol{v}}(1) \bullet \begin{pmatrix} 0\\1 \end{pmatrix}}{\|\vec{\boldsymbol{v}}(1)\| \, \|\begin{pmatrix} 0\\1 \end{pmatrix}\|} = \frac{4\sqrt{2}}{\sqrt{4+32}} = \frac{2}{3}\sqrt{2}.$$

4a $\vec{\boldsymbol{x}}(\theta) = \begin{pmatrix} r\cos\theta\\r\sin\theta \end{pmatrix} = \begin{pmatrix} 2(1-\cos\theta)\cos\theta\\2(1-\cos\theta)\sin\theta \end{pmatrix} = 2\begin{pmatrix} (1-\cos\theta)\cos\theta\\(1-\cos\theta)\sin\theta \end{pmatrix}$

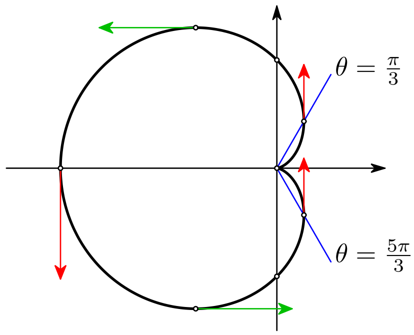
4b On the x-axis one has $\theta = 0$ or $\theta = \pi$, so there are two intersection points: $\theta = 0$, $r = 0$ i.e. the origin, and $\theta = \pi$, $r = 4$, i.e. the point $(-4, 0)$.
On the y-axis one has $\theta = \frac{\pi}{2}$ (and thus $r = 2$) or $\theta = \frac{3\pi}{2}$ (in which case $r = 2$ too), so one again gets two intersections: $(0, \pm 2)$.

4c $\vec{\boldsymbol{x}}'(\theta) = 2\begin{pmatrix} -\sin\theta + 2\cos\theta\sin\theta\\\cos\theta - \cos^2\theta + \sin^2\theta \end{pmatrix}$. This vector is vertical if its horizontal component $(-\sin\theta + 2\cos\theta\sin\theta)$ vanishes **and its vertical component** $(\cos\theta - \cos^2\theta + \sin^2\theta)$ **does not vanish.** (It is easy to forget that last detail, but you have to check that the vertical vector isn't really the zero vector – that would be like the weatherman saying there's a 0mph wind, coming from the north.)

So solve $-\sin\theta + 2\cos\theta\sin\theta = 0$. You get $\sin\theta = 0$, i.e. $\theta = 0$ or $\theta = \pi$, or $\cos\theta = \frac{1}{2}$, which happens for $\theta = \frac{\pi}{3}$ and $\theta = \frac{5\pi}{3}$. So there are four possible vertical points:

$\theta = 0$	$\vec{\boldsymbol{x}}'(0) = \binom{0}{0}$	NOT a vertical point
$\theta = \pi$	$\vec{\boldsymbol{x}}'(\pi) = \binom{0}{-4}$	a vertical point
$\theta = \frac{\pi}{3}$	$\vec{\boldsymbol{x}}'(\frac{\pi}{3}) = \binom{0}{2}$	a vertical point
$\theta = -\frac{\pi}{3}$	$\vec{\boldsymbol{x}}'(-\frac{\pi}{3}) = \binom{0}{2}$	a vertical point

4d Here's the drawing. The problem didn't ask for the horizontal tangents, but if you do look for them you will find them by setting $\frac{dy}{d\theta} = 0$; they are at $\theta = \frac{2\pi}{3}$ and $\frac{4\pi}{3}$ (where $\cos\theta = -\frac{1}{2}$).



5a We must have $\overrightarrow{AB} + \overrightarrow{AC} = \overrightarrow{AD}$. So $\overrightarrow{AD} = \begin{pmatrix} -2 \\ -2 \\ 1+h \end{pmatrix}$ and D is the point $(0, 0, 1 + h)$.

5b The area is $\frac{1}{2}\|\overrightarrow{AB} \times \overrightarrow{AC}\| = \frac{1}{2}\| \begin{pmatrix} 2 \\ -1+h \\ 2 \end{pmatrix} \| = \frac{1}{2}\sqrt{9 - 2h + h^2}$

5c We want $\overrightarrow{AB} \bullet \overrightarrow{BC} = 0$. Compute $\overrightarrow{AB} \bullet \overrightarrow{BC} = h - 1$, so ABC is a right triangle when $h = 1$.

5d The equation is $\vec{\boldsymbol{n}} \bullet \vec{\boldsymbol{x}} = \vec{\boldsymbol{n}} \bullet \vec{\boldsymbol{a}}$, if $\vec{\boldsymbol{n}}$ is a normal to the plane. One such normal is $\vec{\boldsymbol{n}} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} 2 \\ -1+h \\ 2 \end{pmatrix}$ so the equation is $2x_1 + (-1 + h)x_2 + 2x_3 = C$, where C is what you get by substituting $\vec{\boldsymbol{x}} = \vec{\boldsymbol{a}}$ (or $\vec{\boldsymbol{b}}$, or $\vec{\boldsymbol{c}}$, the result should be the same). In this case: $C = 2+2h$, so the equation is $2x_1 + (-1+h)x_2 + 2x_3 = 2+2h$.

5e We found the equation for the plane. The origin will lie on the plane if $(x_1, x_2, x_3) = (0, 0, 0)$ satisfies the equation. Therefore the origin is on the plane if $0 = 2 + 2h$, i.e. $h = -1$.

6a $\vec{\boldsymbol{x}}(2) = \binom{4}{9}$ and $\vec{\boldsymbol{x}}'(2) = \binom{4}{12}$, so the tangent line has parametric representation $\vec{\boldsymbol{y}} = \binom{4}{9} + s\binom{4}{12} = \binom{4+4s}{9+12s}$. This line hits the x_1 axis if $s = -\frac{9}{12} = \frac{3}{4}$, hence at the point $(1, 0)$; it intersects the x_2 axis when $s = -1$, hence at the point $(0, -3)$.

6b The length is $L = \int_{-1}^{1} \|\vec{\boldsymbol{x}}'(t)\|dt$. Since $\|\vec{\boldsymbol{x}}'(t)\| = \sqrt{4t^2 + 9t^4} = |t|\sqrt{4 + 9t^2}$ you get $L = \int_{-1}^{1} |t|\sqrt{4 + 9t^2}dt$. The integrand is even, so you can get rid of the absolute values by using $\int_{-1}^{0} \cdots = \int_{0}^{1} \cdots$: $L = 2\int_0^1 |t|\sqrt{4 + 9t^2}dt$. Next, substitute $u = 4 + 9t^2$ and you get $L = \frac{2}{27}(13^{3/2} - 4^{3/2})$.

7a $\vec{\boldsymbol{n}} = \overrightarrow{AC} \times \overrightarrow{AB} = \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix}$.

7b Area $= \frac{1}{2}\|\overrightarrow{AC} \times \overrightarrow{AB}\| = \frac{1}{2}\sqrt{21}$.

7c D is on $\mathcal{P}$ is $\overrightarrow{AD} \perp \vec{\boldsymbol{n}}$, so compute $\overrightarrow{AD} \bullet \vec{\boldsymbol{n}} = 5h - 7$. Conclusion: D is on $\mathcal{P}$ when $h = \frac{5}{7}$.

8a $\vec{\boldsymbol{x}}'(t) = \binom{t\cos t}{t\sin t}$, so $\vec{\boldsymbol{x}}(\pi/4) = \begin{pmatrix} \frac{1}{2}\sqrt{2}+\frac{\pi}{8}\sqrt{2} \\ \frac{1}{2}\sqrt{2}-\frac{\pi}{8}\sqrt{2} \end{pmatrix}$ and $\vec{\boldsymbol{x}}'(\pi/4) = \begin{pmatrix} \frac{\pi}{8}\sqrt{2} \\ \frac{\pi}{8}\sqrt{2} \end{pmatrix}$. Hence the tangent line has parametric equation

$$\vec{\boldsymbol{z}} = \vec{\boldsymbol{x}}(\pi/4) + s\vec{\boldsymbol{x}}'(\pi/4) = \begin{pmatrix} \frac{1}{2}\sqrt{2}+\frac{\pi}{8}\sqrt{2} \\ \frac{1}{2}\sqrt{2}-\frac{\pi}{8}\sqrt{2} \end{pmatrix} + s\begin{pmatrix} \frac{\pi}{8}\sqrt{2} \\ \frac{\pi}{8}\sqrt{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\sqrt{2}+\frac{\pi}{8}\sqrt{2}+s\frac{\pi}{8}\sqrt{2} \\ \frac{1}{2}\sqrt{2}-\frac{\pi}{8}\sqrt{2}+s\frac{\pi}{8}\sqrt{2} \end{pmatrix}.$$

8b When is $\vec{\boldsymbol{x}}'(t) \perp \binom{1}{-1}$? This happens when $\vec{\boldsymbol{x}}'(t) \bullet \binom{1}{-1} = 0$. So we must solve

$$\vec{\boldsymbol{x}}'(t) \bullet \binom{1}{-1} = t\cos t - t\sin t = 0, \text{ i.e. } t(\cos t - \sin t) = 0.$$

The solutions are either $t = 0$, or $\cos t = \sin t$.

$t=0$ doesn't count, because at $t=0$ you have $\vec{\boldsymbol{x}}'(0)=\vec{\mathbf{0}}$, and the zero vector has no direction so you can't say it is perpendicular to anything.
$\cos t = \sin t$ happens when $t=\pi/4+k\pi$ $(k=0,\pm1,\pm2,\ldots)$

8c The length is

$$\int_0^{\pi/2}\|\vec{\boldsymbol{x}}'(t)\|\mathrm{d}t=\int_0^{\pi/2}\left\|\begin{pmatrix} t\cos t\\ t\sin t\end{pmatrix}\right\|\mathrm{d}t=\int_0^{\pi/2}\sqrt{t^2\cos^2 t+t^2\sin^2 t}\,\mathrm{d}t=\int_0^{\pi/2} t\,\mathrm{d}t=\frac{\pi^2}{8}.$$

9a We want to know when $\overrightarrow{AC}\bullet\overrightarrow{BC}=0$? Compute the dot product to find: $\overrightarrow{AC}\bullet\overrightarrow{BC}=q^2-q+2$. This is positive for all q, so the angle at C is never a right angle. Since the dot product $\overrightarrow{AC}\bullet\overrightarrow{BC}=q^2-q+2$ is always positive, the angle at C is always an acute angle.

9b A normal to the plane is given by $\vec{\boldsymbol{m}}=\overrightarrow{AB}\times\overrightarrow{AC}=\begin{pmatrix}4\\ -2-q\\ -2q\end{pmatrix}$. The equation of the plane is therefore $4x_1-(2+q)x_2-2qx_3=-4q$.

9c If $(0,8,0)$ satisfies the defining equation we just found then $-(2+q)8=-4q$. This happens for $q=-4$.

9d The distance from the origin O to the plane $\mathcal{P}$ is (up to a sign) $\frac{\vec{\boldsymbol{m}}\bullet(\vec{\mathbf{0}}-\vec{\boldsymbol{a}})}{\|\vec{\boldsymbol{m}}\|}$, with $\vec{\boldsymbol{m}}$ as in part (b) of this problem. You find $\|\vec{\boldsymbol{m}}\|=\sqrt{116}=2\sqrt{29}$ and $\vec{\boldsymbol{m}}\bullet\vec{\boldsymbol{a}}=-16$ so that the distance is $\frac{-16}{2\sqrt{29}}=-\frac{8}{29}\sqrt{29}$. The minus sign indicates that the origin lies on one side of the plane $\mathcal{P}$, and the normal $\vec{\boldsymbol{m}}$ which we have used points to the other side of $\mathcal{P}$.

10a $\vec{\boldsymbol{x}}(2)=\begin{pmatrix}4\\ -6\end{pmatrix}$, $\vec{\boldsymbol{x}}'(2)=\begin{pmatrix}4\\ -11\end{pmatrix}$, so a parametric equation for the tangent to the curve at $t=2$ is $\vec{\boldsymbol{y}}=\begin{pmatrix}4\\ -6\end{pmatrix}+s\begin{pmatrix}4\\ -11\end{pmatrix}=\begin{pmatrix}4+4s\\ -6-11s\end{pmatrix}$.

10b The tangent line at $\vec{\boldsymbol{x}}(t)$ is horizontal if the velocity vector $\vec{\boldsymbol{x}}'(t)$ is horizontal. This happens exactly if the vertical component of vx' vanishes, while the horizontal component of $\vec{\boldsymbol{x}}'$ does not. So we have to solve

$$x'(t)=2t\neq 0,\ \text{and}\ y'(t)=1-3t^2=0.$$

The solutions are $t=\pm\frac{1}{3}\sqrt{3}$. The corresponding points have position vectors $\vec{\boldsymbol{x}}(\pm\frac{1}{3}\sqrt{3})=\begin{pmatrix}\frac{1}{3}\\ \pm\frac{2}{9}\sqrt{3}\end{pmatrix}$.

10c The length is $L=\int_{-1}^{1}\|\vec{\boldsymbol{x}}'(t)\|dt=\int_{-1}^{1}\sqrt{4t^2+(1-3t^2)^2}\,dt$. This integral can't be done with the methods and functions we have learned in 221/222.