TOPICS COVERED IN THE SECOND MIDTERM, MATH 522, SPRING 2009

(1) Differentiation inside an integral: when can we differentiate wrt t the integral of a function $\phi(x,t)$, if we integrate wrt x? Notice that I did a simplification of the Theorem in the book, so it is better to look at the notes. As usual, you will need to check hypothesis if you apply the Theorem.
(2) Definition of integral.
   (a) Definition of the integral of a continuous function over k-cells (as iteration of integrals in one variable).
   (b) Theorem showing that the integral of continuous functions over k-cells does not depend on the order of integration (using Stone-Weierstrass).
   (c) Definition of integral over other compact domains (by extending it trivially to a cell containing the domain). Theorem showing that for these discontinuous functions the integral is still well defined (using the "glue" function).
(3) Change of variable formula.
   (a) Definition of primitive functions and flips.
   (b) Theorem stating that a C^1 map between open set of $\mathbb{R}^n$ with $F(0) = 0$ and $F'(0)$ invertible, can be decomposed into composition of primitive functions and flips (the fancy "row reduction" process).
   (c) Definition of partition of unity associated to an open cover of a compact set.
   (d) Change of variable Theorem. Learn the steps in the proof (how do we apply all the previous concepts in this proof?) Be ready to prove this theorem in special cases (for example, when T is a primitive function).
(4) Differential forms and surfaces: their definitions, algebra of k-forms (how do you add, multiply by functions and elementary properties).
(5) Basic k-forms and the Theorem showing that if a form is written in terms of basic forms, it is zero iff each coefficient is zero.
(6) The wedge product and how to use it. The differential and how to apply it to k-forms. Properties of the differential ($d^2 = 0$ and the product rule), including the proof of why $d^2 = 0$, which I could ask.
(7) Stokes theorem.
   (a) Effect of a pullback in forms, in their wedge product and in their differentiation.
   (b) Effect of the composition of two pullbacks in the forms and their wedge product. Learn well how pullbacks work.
   (c) Effect of pullbacks in the integral of k-forms.
   (d) Affine simplexes and their orientation. Effect of a change of orientation on the integral over the simplex. The oriented boundary of an affine simplex. Affine chains and boundaries of affine chains.
   (e) Differentiable Chains and their boundaries. Be sure to know examples.
   (f) Statement of Stokes Theorem, and how to use it.

PRACTICE PROBLEMS

This list is not exhaustive, it is meant only for practice and to give an idea of the level of the questions.

(1) Consider the function
$$f(\lambda, x) = \int_0^x \frac{\sin(\lambda u)}{u} du.$$
Calculate $\frac{\partial f}{\partial \lambda}(\lambda, x)$. If you are going to apply any Theorem, be sure to check its hypothesis.
(2) Suppose that a function $\phi : [a,b] \to \mathbb{R}$ is continuous. Prove that
$$2\int_a^b \left[f(x) \int_x^b f(y)dy \right] dx = \left[\int_a^b f(x)dx \right]^2.$$

(3) Let $= \{(u,v), u > 0, v > 0\}$ and define $\Phi : U \to \mathbb{R}^2$ by $\Phi(u,v) = (u^2 - v^2, uv)$. These are called *hyperbolic coordinates*. Let $\mathcal{S} = \Phi([1,9] \times [2,4])$. Calculate

$$\int_{\mathcal{S}} (x^2 + y^2) dx dy.$$

(Do not worry about the precise number you get).

(4) If $\omega = f$ is a zero form and $\lambda = g dx_J$ where dx_J is a basic k-form, prove that

$$d(\omega \wedge \lambda) = d\omega \wedge \lambda + \omega \wedge d\lambda.$$

(5) If Φ is a differentiable chain with zero boundary in an open set $V \subset \mathbb{R}^m$, and if θ is an $n-1$ form on V, show that $d\theta$ must vanish at some point $\mathbf{x} \in V$.

(6) Let Φ be the differentiable chain given by

$$\Phi : [1,2] \times [0, 2\pi] \to \mathbb{R}^2, \quad \Phi(r, x) = (r\cos\theta, r\sin\theta).$$

Find the oriented boundary of Φ, and be sure to include a drawing. Let $\omega = xdy + ydx$. Calculate

$$\int_{\partial\Phi} \omega.$$

Same question for $\widehat{\omega} = xdy - ydx$.

(7) Let Φ be a surface defined by

$$\Phi : [0,1] \times [0,1] \to \mathbb{R}^3, \quad \Phi(t,s) = (g(t)h_1(s), g(t)h_2(s), g(t)h_3(s))$$

where g, h_1, h_2, h_3 are all $C^2([0,1])$. Let

$$\xi = xdy \wedge dz + ydz \wedge dx + zdx \wedge dy.$$

(a) Calculate

$$\int_{\Phi} \xi$$

without using Stokes Theorem.

(b) Is there a solid V in $\mathbb{R}^3$ with nonzero volume such that $\partial V = \Phi$, where ∂V is the positively oriented boundary of V? Explain.

(8) Consider the differentiable chain (a torus) $\sigma : [0, 2\pi] \times [2, \pi] \to \mathbb{R}^3$

$$\sigma(u,v) = ((a + b\cos u)\cos v, (a + b\cos u)\sin v, b\sin u).$$

Calculate $\partial\sigma$ and prove that, if ω is any C^2 1-form, then

$$\int_{\sigma} \omega = 0.$$