PRACTICE PROBLEMS, MATH 522, SPRING 2009

(1) Let ω be a form of odd degree on U and let f be a function such that $f(x) \neq 0$ for any $x \in U$, and such that $d(f\omega) = 0$. Show that
$$\omega \wedge d\omega = 0$$
Hint: this is not as trivial as it looks, in general $\omega \wedge \omega \neq 0$ (take $\omega = dx_1 \wedge dx_2 + dx_3 \wedge dx_4$. Then $\omega \wedge \omega = 2dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4$). Try to calculate $d(f\omega \wedge \omega)$ in two different ways - placing f in two different places - to prove this.

(2) Assume ω_1 and ω_2 are exact forms. Show that $\omega_1 \wedge \omega_2$ is also exact.

(3) Assume that ω is a 1-form in $\mathbb{R}^3 - \{\mathbf{0}\}$, of class C^1 and $d\omega = 0$. Show that ω is exact in $\mathbb{R}^3 - \{\mathbf{0}\}$.

(4) Assume ω is a closed form, C^1 on a convex domain E. Assume Φ_1 and Φ_2 are two surfaces such that their positively oriented boundaries coincide. Show that
$$\int_{\Phi_1} \omega = \int_{\Phi_2} \omega$$
that is, the integral of ω depends only on its value on the boundary.

(5) Given a curve $\gamma = (\gamma_1, \gamma_2)$ in $\mathbb{R}^2$, define the positively oriented unit normal to γ as the vector $\mathbf{n} = \frac{1}{\|\gamma'\|}(\gamma_2', -\gamma_1')$. Let ∇ represent the gradient and $\Delta f = D_{11}f + D_{22}f$ the Laplacian of a function f. Assume that Φ is a surface in $\mathbb{R}^2$ and γ is its positively oriented boundary. Show that
$$\int\int_{\Phi} (g\Delta f - f\Delta g)\, dxdy = \int_{\gamma} (g\nabla f \cdot \mathbf{n} - f\nabla g \cdot \mathbf{n})\, ds.$$

(6) I don't think I can come up with more of these Green/Stokes/Divergence types of problems. I hope you have enough!