PRACTICE PROBLEMS, MATH 522, SPRING 2009

(1) Suppose that the functions $\phi : \mathbb{R} \to \mathbb{R}$ and $\xi : \mathbb{R} \to \mathbb{R}$ have continuous second order partial derivatives. Define

$$f(x,y) = \phi(x-y) + \xi(x+y)$$

for $(x,y) \in \mathbb{R}^2$. Prove that

$$\frac{\partial^2 f}{\partial x^2}(x,y) - \frac{\partial^2 f}{\partial y^2}(x,y) = 0$$

for any $(x,y) \in \mathbb{R}^2$.

(2) Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} x^2y^4/(x^2+y^2) & if\ (x,y) \neq (0,0) \\ 0 & if\ (x,y) = (0,0). \end{cases}$$

prove that f has first order partial derivatives. Is the function continuously differentiable?

(3) Suppose that $F : \mathbb{R}^2 \to \mathbb{R}^2$ has the property

$$\|F(x,y)\| \leq x^2 + y^2$$

for every $(x,y) \in \mathbb{R}^2$. Show that

$$\lim_{(x,y)\to(0,0)} \frac{\|F(x,y)\|}{\|(x,y)\|} = 0.$$

Prove that F is differentiable at $(0,0)$ and its differential vanishes.

(4) A twice differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ is called *harmonic* whenever $\sum_{k=1}^{n} \frac{\partial^2 f}{\partial x_k^2} = 0$.

Let U be the open set given by $\|\mathbf{x}\| > 0$ and let f be given by

$$f(\mathbf{x}) = \frac{1}{\|\mathbf{x}\|}.$$

Prove that f is harmonic.

(5) Assume $f(x,y)$ is harmonic and let $g(x,y) = f(x^2-y^2, 2xy)$. Prove that g is also harmonic.

(6) Given two real numbers a and b, consider the system of equations

$$\begin{aligned} x + x^2\cos y + xye^{x^3y^2} &= a \\ y + x^5 + y^3 - x^2\cos(xy) &= b. \end{aligned}$$

Use the Inverse Function Theorem to show that there exists $r > 0$ such that the equation has a solution whenever $a^2 + b^2 < 1$.

(7) Use the Implicit Function Theorem to analyze the existence of solutions of the system

$$\begin{aligned} (uv)^4 + (u+s)^3 + t &= 0 \\ \sin(uv) + e^{v+t^2} - 1 &= 0 \end{aligned}$$

near the zero solution, where $(x,y,z,t) \in \mathbb{R}^4$.