ON COMPLETELY INTEGRABLE GEOMETRIC EVOLUTIONS
OF CURVES OF LAGRANGIAN PLANES

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ABSTRACT. In this paper we find a explicit moving frame along curves of Lagrangian planes invariant under the action of the symplectic group. We use the moving frame to find a family of independent and generating differential invariants. We then construct geometric Hamiltonian structures in the space of differential invariants and prove that, if we restrict them to a certain Poisson submanifold, they become a set of decoupled KdV first and second Hamiltonian structures. We find an evolution of curves of Lagrangian planes that induces a system of decoupled KdV equations on their differential invariants (we call it the Lagrangian Schwarzian KdV equation). We also show that a generalized Miura transformation takes this system to a modified matrix KdV equation. In the four dimensional case we also find two Nijenhuis operators associated to the unrestricted geometric Poisson brackets.

1. INTRODUCTION

In [M1] the author described a general method to generate compatible Hamiltonian structures from the geometry of curves in flat homogeneous spaces of the form \(G/H\), where \(G\) is a semisimple Lie group. The method relied on the use of a less-known concept of moving frame as an equivariant map from the jet space into the group. In some cases one can identify the pull back of the Maurer Cartan form in the group via a moving frame with the set of differential invariants (curvature and torsion in the Euclidean case) of the curves. The author called this the space of moving coframes. The Hamiltonian structures were obtained by reduction of well-known Hamiltonian structures in \(Lg^*\) to the space of moving coframes. The reduction was done in such a way that one could determine which geometric evolutions of curves in \(G/H\) will induce a Hamiltonian flow on its differential invariants. Even though the description in [M1] is general, the construction assumes knowledge of the moving frame associated to the different geometries (in the Klein sense) of curves. Unfortunately moving frames are not known in general.

In this paper we first study the geometric background of curves of Lagrangian planes as curves in the homogeneous space \(Sp(2n)/H\), where \(H\) is the isotropy subgroup of a distinguished point. This quotient can be identified with the space of \(n \times n\) symmetric matrices and also with the space of Lagrangian planes in standard symplectic \(\mathbb{R}^{2n}\) (what is usually called the Grassmann Lagrangian manifold). In section 3 we first find a moving frame along curves in \(Sp(2n)/H\) and we use it to classify all differential invariants for the curves under the action of the entire \(Sp(2n)\) group. Notice that [Ov] classifies the so-called projective invariants of Lagrangian

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planes. As we will see, the projective invariants are only a part of the invariants presented here and the invariant manifold splits linearly into projective and non-projective invariants. See also [AD]. We then use the moving coframe associated to our frame to produce several pairs of compatible Poisson brackets associated to evolutions of curves of Lagrangian planes invariant under the Sp(2n) (this is done in section 4). Following [M1] we describe which geometric evolutions will induce Hamiltonian flows on their invariants.

In section 4 we also describe geometric evolutions of Lagrangian planes whose invariantization, that is, the evolution induced on the differential invariants of the flow, is completely integrable and given by a set of decoupled KdV equations. The evolution is a generalization of the Schwarzian KdK equation to the case of Lagrangian planes. These flows have vanishing non-projective invariants. If one considers generic differential invariants, then we obtain families of compatible Poisson brackets that can be used for integration. For example, in the special case \( n = 2 \) we produce two pairs of compatible Hamiltonian bracket, an invertible one in each pair, which produce two Nijenhuis operators. It is not clear if these operators are associated to some integrable system, but we give the condition on the curve evolution that guarantees that the invariantization of the evolution is Hamiltonian.

2. Definitions and description of previous work

2.1. Moving frames, coframes and classification of differential invariants. Let \( G \) be a Lie group acting on a manifold \( M \) transitively. Then \( M \) can be locally identified with the quotient \( G/H \) where \( H \) is the isotropy subgroup of some distinguished point \( o \). If \( G/H \) is flat and \( G \) is semisimple, it is known ([Oc]) that its Lie algebra \( g \) can be split into

\[
g = g_1 \oplus g_0 \oplus g_{-1}
\]

where \( g_1 \) and \( g_{-1} \) are dual to each other with respect to the Killing form and where \( g_0 \) leaves \( g_1 \) and \( g_{-1} \) invariant under the adjoint action.

Let \( J^k(M, \mathbb{R}) \) the \( k \)-jet space of curves, that is, the set of equivalence classes of curves in \( M \) up to \( k \)th order of contact. If we denote by \( u(x) \) a curve in \( M \) and by \( u_r \) the \( r \) derivative of \( u \) with respect to the parameter \( x \), \( u_r = \frac{d^r u}{dx^r} \), the jet space has local coordinates that can be represented by \( u^{(k)} = (x, u, u_1, u_2, \ldots, u_k) \). The group \( G \) acts naturally on parametrized curves, therefore it acts naturally on the jet space via the formula

\[
g \cdot u^{(k)} = (x, g \cdot u, (g \cdot u)_1, (g \cdot u)_2, \ldots)
\]

where by \((g \cdot u)_k\) we mean the formula obtained when one differentiates \( g \cdot u \) and then write the result in terms of \( g, u, u_1, \) etc. This is usually called the prolonged action of \( G \) on \( J^k(M, \mathbb{R}) \).

Definition 1. A function

\[
I : J^k(M, \mathbb{R}) \to \mathbb{R}
\]

is called a \( k \)th order differential invariant if it is invariant with respect to the prolonged action of \( G \).

Definition 2. A map

\[
\rho : J^k(M, \mathbb{R}) \to G
\]
is called a left (resp. right) moving frame if it is equivariant with respect to the prolonged action of $G$ on $J^k(M,\mathbb{R})$ and the left (resp. right) action of $G$ on itself. The pull-back of the Maurer-Cartan form by a moving frame is the so-called moving coframe. It contains in its entries a complete set of differential invariants as we will see in our particular case. The moving coframe is the equivalent to the Euclidean Frenet equations in this group-based environment.

In [FO1, FO2] the authors recently described a simple method of normalization that can be used to find moving frames explicitly. They also classified all differential invariants and showed how they were generated by the moving frame. The idea of the normalization method is that choosing enough normalization equations of the form

\[(g \cdot u)_r = c_r.\]

The element $g$ will be determined uniquely in terms of $u$ and its derivatives $u_k$. Such $g$ evaluated along a curve will be a moving frame. See [FO1, FO2] for more details.

**Theorem 1.** If the $G$ action is locally effective and $k$ is high enough, then there exists a $k^{th}$ order right moving frame $\rho$ that can be determined uniquely using normalization equations of the form (2.2).

**Definition 3.** The fundamental $n$th order normalized differential invariants associated to a moving frame $\rho$ of order $n$ or less are given by

\[(2.3) I^{(n)}(u^{(n)}) = \rho(u^{(n)}) \cdot u^{(n)}.\]

Assume that we have used a certain number of entries of $(g \cdot u)_r = c_r$ for normalization and we have uniquely determined a moving frame $\rho$ satisfying those equations. Then the following theorem classifies all invariants. Notice that, since $\rho$ satisfies the normalization equations, some of the entries of the normalized differential invariants might be constant. We call those phantom invariants.

**Theorem 2.** A generating system of differential invariants consists of

1. all non-phantom zeroth order differential invariants, and
2. all non-phantom differential invariants of the form $I^{(n+1)}_\alpha$, where $I^{(n)}_\alpha$ is a phantom differential invariant, where $\alpha$ indicates the component, that is, $u = (u^\alpha) \in M$.

That is, any other differential invariant can be written as a function of the ones above and their derivatives with respect to $x$.

2.2. Geometric Hamiltonian structures. Given a semisimple Lie group $G$ there exist natural families of Poisson brackets defined on the space of loops on the dual of its Lie algebra, $Lg^*$. They can be defined as follows: let $H : Lg^* \to \mathbb{R}$ be a functional defined on $Lg^*$. To $H$ we associate an element of $Lg^*$, $\delta H \over \delta L$, determined by the Frechet derivative

\[\lim_{\epsilon \to 0} H(L + \epsilon v) = \langle v, \delta H \over \delta L(L) \rangle.\]

The element $\delta H \over \delta L(L)$ is called the variational derivative of $H$ at $L$. 

Given two functionals on $Lg^*$ we can define their Kac-Moody Lie-Poisson bracket as the bracket given by the relation
\begin{equation}
\{H, G\}(L) = \langle \delta G \delta L, \left( \frac{\delta H}{\delta L} \right)_x + \text{ad}^* \left( \frac{\delta H}{\delta L} \right)(L) \rangle
\end{equation}
where here we are identifying $g$ and $g^*$ using the nondegenerate Killing form. This bracket is well-known to be a Poisson bracket in the space of functionals on $Lg^*$. Further more, its symplectic leaves (the leaves where Hamiltonian flows lie) coincide with the orbits in $Lg^*$ under a Kac-Moody action of the group $LG$ on $Lg^*$. This action is given by
\begin{equation}
A(g)(L) = g^{-1} g_x + \text{Ad}^*(g)L.
\end{equation}

There is an additional family of simpler Poisson brackets defined on $Lg^*$. Given two functionals on $Lg^*$ we define their Poisson bracket by the formula
\begin{equation}
\{H, G\}_0(k) = \langle \delta G \delta L, \text{ad}^*(L_0) \left( \frac{\delta H}{\delta L} \right) \rangle
\end{equation}
where $L_0 \in g^*$ is any constant element. These families of Poisson brackets are all known to be compatible, that is, any linear combination of these brackets is also a Poisson bracket.

The definition of geometric Poisson brackets is based on the following fact: one can describe the set of moving coframes as a quotient of a submanifold of $Lg^*$ by the Kac-Moody action (2.5) of a properly chosen isotropy subgroup of $LG$. This implies that the bracket (2.4) can be reduced to the space of coframes and one can easily check that the brackets (2.6) are also reducible. We call these reductions geometric Poisson brackets.

Geometric Poisson brackets are found explicitly as follwos. First of all, assume that $\rho$ is a left moving frame and $\rho = \rho_{-1} \rho_0 \rho_1$ is he local factorization induced by the splitting (2.1) (locally $G = G_{-1} \cdot G_0 \cdot G_1$ where $G_i$ is the subgroup corresponding to $g_i$). Let $K = \rho^{-1} \rho_x = K_{-1} + K_0 + K_1$ be the splitting induced on the associated moving coframe. The following theorems can be found in [M1].

**Theorem 3.** There exists a left moving frame $\rho$ such that $K_{-1} = \Lambda$ is constant and $\rho_{-1}$ can be identified with $u$.

It is known that the adjoint action of $G_0$ on $g_{-1}$ is linear. Hence, any element in $G_0$ can be identified with an element in $GL(n, \mathbb{R})$ in that sense. The following theorem can be found in [M1].

**Theorem 4.** Let $\rho$ be a left moving frame and assume $\rho = \rho_{-1} \rho_0 \rho_1$ as above. If $\rho_0$ is identified with an element of $GL(n, \mathbb{R})$, then $\rho_0$ contains in columns a classical moving frame along the curve $u$, that is an invariant curve in the tangent bundle along the curve.

Let $M \subset Lg^*$ be the submanifold generated by loops with values in $g_0 \oplus g_1 \oplus \{\Lambda\}$ and with positive $\Lambda$-component. Let $N_0 \subset G_0$ be the isotropy subgroup of $\Lambda$ in $G_0$ and let $N = L(G_1 \cdot N_0)$, where $\cdot$ represents the semidirect product.

**Theorem 5.** There exists an open set $U \subset M$ such that $N$ acts on $U$ with action (2.5), and such that $U/N$ can be identified with $K_0$, the space of moving coframes. Furthermore, the Poisson bracket (2.4) can be Poisson reduced to $U/N$. 
One of the most interesting things about defining these brackets using reduction is that one can explicitly find the reduced bracket in specific examples as we will see in the next section. A theorem in [M1] states that the brackets (2.6) can also be easily reduced to $U/N$ whenever $L_0 \in \mathfrak{g}_1$.

Finally, one other great advantage of reducing (2.4) is that the associated reduced Hamiltonian flow can be readily related to geometric evolutions of curves, that is, evolutions of curves for which the group $G$ takes solutions to solutions. Geometric evolutions are those of the form

$$u_t = Fr = r_1 F_1 + r_2 F_2 + \cdots + r_n F_n$$

where $F = (F_1, F_2, \ldots, F_n)$ is an invertible matrix containing in columns a classical moving frame along $u$ and where $r = (r_1, \ldots, r_n)^T$ is a vector whose entries are differential invariants of $u$.

**Theorem 6. ([M1])** Let $u(t, x)$ be a family of curves solution of a geometric evolution of the form (2.7). Let $v_r \in \mathfrak{g}_{-1}$ be determine by $r$ if we left-identify $\mathfrak{g}_{-1}$ with the tangent to $G/H$. Assume that there exists a local functional $h : K \to \mathbb{R}$ and a local extension $H : M \to \mathbb{R}$ constant on the orbits of $N$. Assume further than

$$\frac{\delta H}{\delta L} = H_{-1} + H_0 + H_1$$

is the splitting induced by (2.1) with $H_{-1} = v_r$. Let $k$ be the differential invariants defined by $K = \rho^{-1} \rho_x$, where $\rho = \rho^{-1} \rho_0 \rho^{-1}$, $\rho_{-1}$ is identified with $u$ and $\rho_0$ is determined by $F$ as in Theorem 4. Then, the evolution induced by (2.7) on $k$ is Hamiltonian with respect to the reduction of (2.4) and its associated Hamiltonian is $h$.

Although this is a very brief description of the definition of geometric bracket and its relation to geometric evolutions, the next chapter will find the needed information to fill the gaps that were previously unknown in the Lagrangian Grassmanian case and will apply the reduction to find Hamiltonian structures on the space of differential invariants of curves of Lagrangian planes. For more details on geometric Poisson brackets, please see [M1].

3. A MOVING FRAME FOR CURVES OF LAGRANGIAN PLANES

Since this is a local study of curves we will assume to be close enough to the identity in $Sp(2n)$, enough to be sure that any matrix in $Sp(2n)$ can be factored as $A = A_1 A_0 A_{-1}$ following the splitting (2.1). That is

$$A = \begin{pmatrix} I & 0 \\ \hat{S} & I \end{pmatrix} \begin{pmatrix} g & 0 \\ 0 & g^{-T} \end{pmatrix} \begin{pmatrix} I & S \\ 0 & I \end{pmatrix} = A_1 A_0 A_{-1}$$

where $I$ is the identity matrix, both $S$ and $\hat{S}$ are symmetric matrices and $g \in GL(n, \mathbb{R})$. If we assume this factorization, then $H = G_1 \cdot G_0$ is given by the elements with only the first two factors and the homogeneous space $Sp(2n)/H$ can be readily identified with the space of symmetric matrices. This homogeneous space is also identified with the manifold of Lagrangian planes or Grassmann Lagrangian manifold. If $u \in Sp(2n)/H$ is a symmetric matrix, the action of $Sp(2n)$ on the homogenous space is determined by the relation

$$A \begin{pmatrix} I & u \\ 0 & I \end{pmatrix} = \begin{pmatrix} I & A \cdot u \\ 0 & I \end{pmatrix}$$
where $A$ is as in (3.1) and where $h \in H$. This formula defines the action

\begin{equation}
A \cdot u = g(u + S) \left( g^{-T} + \hat{S}g(u + S) \right)^{-1}.
\end{equation}

In order to find a right moving frame, that is, an equivariant map from the jet space to the group, we use normalization equations of the type (2.2) to determine it, following Fels and Olver’s normalization procedure.

For simplicity we will assume that $u_1$ is positive definite. This, of course, needs not to be true. But this is a local study and other cases can be modified accordingly and produce similar results.

\textbf{Theorem 7.} Let $u(x) \in \text{Sp}(2n)/H$ be a curve of Lagrangian planes and let $u_k = \frac{d^k u}{dx^k}$. A left invariant moving frame $\rho$ is given by

\begin{equation}
\rho = 
\begin{pmatrix}
1 & u \\
0 & I
\end{pmatrix}
\begin{pmatrix}
\frac{u_1^{1/2} \Theta^T}{u_1^{1/2} \Theta^T} & 0 \\
0 & u_1^{-1/2} \Theta^T
\end{pmatrix}
\begin{pmatrix}
I \\
-\frac{1}{2} \Theta u_1^{-1/2} u_2 u_1^{-1/2} \Theta^T & I
\end{pmatrix}
\end{equation}

where the matrix $\Theta$ is the element of $O(n)$ determined by the Gramm-Schmidt process when diagonalizing the Lagrange Schwarzian derivative

\begin{equation}
S(u) = u_1^{-1/2} \left( u_1 - \frac{3}{2} u_2 u_1^{-1} u_2 \right) u_1^{-1/2}.
\end{equation}

That is, $\Theta$ is an element in $O(n)$ satisfying

\[ \Theta S(u) \Theta^T = D \]

for some diagonal matrix $D$.

The Lagrange Schwarzian derivative was first introduced by V. Ovsienko in [Ov]. See [OT] for a more complete description and [AD] for a study of the projective geometry of Lagrangian planes.

\textbf{Proof.} The normalization procedure described in [FO1, FO2] will produce a right invariant moving frame and the one given in the statement of the theorem will be its inverse. Assume the right invariant frame can be factored as in (3.1). We can choose zero order normalization equations of the form

\[ A \cdot u = g(u + S) \left( g^{-T} + \hat{S}g(u + S) \right)^{-1} = 0 \]

which can be solved by choosing $S = -u$. The first order normalization equations are obtained by differentiating the previous one. If we call $M = g^{-T} + \hat{S}g(u + S)$ they are given by

\[ gu_1 M^{-1} - g(u + S)M^{-1} \hat{S}gu_1 M^{-1} = I \]

which, after substituting the values found in the previous normalization, becomes

\begin{equation}
gu_1 g^T = I.
\end{equation}

The equation above can be solved choosing $g$ of the form $g = \Theta u_1^{-1/2}$, where $\Theta$ is some element of $O(n)$ to be determined by further normalizations. The symmetric matrix $u_1^{1/2}$ is defined as in [Ov]. We now go into the second order normalization equations. They are given by

\begin{align*}
gu_2 M^{-1} - 2gu_1 M^{-1} \hat{S}gu_1 M^{-1} \\
-g(u + S) \left[-2M^{-1} \hat{S}gu_1 M^{-1} \hat{S}gu_1 M^{-1} + M^{-1} \hat{S}gu_2 M^{-1} \right] = 0
\end{align*}
which, after substituting previous normalizations, becomes
\[ gu_2g^T - 2\hat{S} = 0. \]
This determines the value \( \hat{S} = \frac{1}{2}gu_2g^T \). Finally, the third order normalizations are similarly obtained and, after incorporating previous normalization results, we need to normalize some entries of
\[
(3.6)
g \left( u_3 - \frac{3}{2}u_2g^T gu_2 \right) g^T.
\]
If we write \( g \) in terms of \( \Theta \) we get that (3.6) is equal to
\[
\Theta S(u)\Theta^T,
\]
where \( S(u) \) is the Lagrange Schwarzian derivative (3.4). This expression can be made diagonal by choosing \( \Theta \) to be the element of \( O(n) \) guaranteed by the Gramm-Schmidt process. This completes the description of the right invariant moving frame. Inverting it produces the one described in the statement of the theorem. ♣

Notice that, as we explained in the previous section, the \( G^{-1} \) component of \( \rho \),
\[
\rho^{-1} = \begin{pmatrix} I & u \\ 0 & I \end{pmatrix}
\]
can be identified with \( u \) itself. Also, it was shown in [M1] that the \( \rho_0 \) factor of the moving frame
\[
\rho_0 = \begin{pmatrix} u_1^{\frac{4}{5}}\Theta^T & 0 \\ 0 & u_1^{-\frac{4}{5}}\Theta^T \end{pmatrix}
\]
can be identified with a classical frame (a curve in the tangent bundle along \( u \)). In fact ([Oc]) \( G_0 \) acts linearly on \( M = G/H \) and, as an element of \( GL(n, \mathbb{R}) \) with \( n = \dim M \), it contains in columns elements of the tangent bundle along \( u \). These vectors were proved to form a classical moving frame in [M1]. In this case, the action of \( \rho_0 \) on any \( y \) is given by \( u_1^{1/2}\Theta^T y\Theta u_1^{1/2} \). We will come back to this point later.

**Theorem 8.** A complete set of independent and generating differential invariants for Lagrangian curves is given by the eigenvalues of the Lagrange Schwarzian derivative plus the strictly upper triangular entries of the matrix
\[
(3.7) \quad I_2 = \Theta^T u_1^{-1/2} \left[ u_4 - 2u_3u_1^{-1}u_2 - 2u_2u_1^{-1}u_3 + 3u_2u_1^{-1}u_2u_1^{-1}u_2 \right] u_1^{-1/2}\Theta
\]
where \( \Theta \) is as in the previous theorem.

**Proof.** The proof of this theorem is the direct application of Fels and Olver’s classification of invariants given in theorem 2. All zero, first and second order invariants are ghost invariants, so the first group of independent differential invariants are the ones appearing in the third order normalized invariants. These are the eigenvalues of the Lagrange Schwarzian derivative since we chose \( \Theta \) to be the unique element such that (3.6), the expression for the 3rd order normalized invariant, is diagonal given by \( D \). Generically all eigenvalues will be independent.

But the off-diagonal entries of (3.6) are also ghost invariants. Expression (3.7) is actually found to be the fourth order normalized invariants. Hence, for example, the strictly upper triangular entries of (3.7) will also be independent differential invariants.
invariants. According to Theorem 2 this group of invariants generates all other differential invariants.

**Theorem 9.** The moving coframe given by the pull-back of the Maurer-Cartan form of the symplectic group by the moving frame $\rho$ is of the form

\[(3.8)\]

\[
K = \begin{pmatrix} K_0 & I \\ K_1 & K_0 \end{pmatrix}
\]

where $K_0$ is skew symmetric and $K_1 = -\frac{1}{2} D$ is diagonal with $D$ being the diagonalization of the Lagrange Schwarzian derivative. The entries of $K_0$ and $K_1$ form a set of independent and generating differential invariants for curves of Lagrangian planes under the action of the symplectic group.

*Note:* The reader can compare this moving coframe to the one associated to projective curves under the action of $SL(2)$. In this case the moving coframe is given by

\[
\begin{pmatrix} 0 & 1 \\ -\frac{1}{2} S(u) & 0 \end{pmatrix}
\]

where $S(u) = \frac{u_3 u_1 - 3/2 u_2^2}{u_1^2}$ is the standard Schwarzian derivative of curves.

*Proof.* The proof of this theorem is based on a simple calculation. Indeed, if $\rho = \rho_1 \rho_0 \rho_1$ is the factorization given in (3.3), then

\[
\rho^{-1} \rho_x = \rho_1^{-1} (\rho_1)_x + \rho_1^{-1} \rho_0^{-1} (\rho_0)_x \rho_1 + \rho_1^{-1} \rho_0^{-1} \rho_1^{-1} (\rho_1)_x \rho_0 \rho_1.
\]

The last term is easily seen to be the identity matrix (in fact, it will always be the normalized value appearing in (3.5). See [M1]). The first term is given by

\[
\rho_1^{-1} (\rho_1)_x = \begin{pmatrix} 0 & 0 \\ \frac{1}{2} (\Theta u_1^{-1/2} u_2 u_1^{-1/2} \Theta^T)_x \\ 0 \end{pmatrix},
\]

and the second by

\[
\rho_1^{-1} \rho_0^{-1} (\rho_0)_x \rho_1 = \begin{pmatrix} 0 \\ \Theta u_1^{-1/2} u_1^{1/2} \Theta^T \\ F \end{pmatrix} \Theta u_1^{-\frac{1}{2}} \left( u_1^{1/2} \Theta^T ight)_x,
\]

where $F = \frac{1}{2} \left[ \Theta u_1^{-1/2} u_2 u_1^{-1/2} \Theta^T, \Theta u_1^{-\frac{1}{2}} \left( u_1^{1/2} \Theta^T \right)_x \right]$.

Now notice that, if $g = \Theta u_1^{-\frac{1}{2}}$

\[
g u_1 g^T = I
\]

and so, differentiating this equation we get

\[
g_x g^{-1} + g^{-T} g_x^T = -g u_2 g^T.
\]

Using this relation we can first conclude that the diagonal block in the moving coframe, $K_0 = -g_x g^{-1} - \frac{1}{2} g u_2 g^T$, is skew symmetric. Also, we can use it to rewrite the lower left block of the moving coframe as

\[
K_1 = \frac{1}{2} (\Theta u_1^{-1/2} u_2 u_1^{-1/2} \Theta^T)_x + F = \frac{1}{2} g \left( u_3 - \frac{3}{2} u_2 u_1^{-1} u_2 \right) g^T
\]

which, with the choice of $g$ given previously, becomes diagonal with entries given by $-\frac{1}{2}$ the diagonalization of the Lagrange Schwarzian derivative.
Finally, given that the diagonal block is
$$K_0 = -g_x g^{-1} - \frac{1}{2} g u_2 g^T,$$
if we differentiate
$$g(u_3 - \frac{3}{2} u_2 u_1^{-1} u_2) g^T = D$$
we get, after a short rewriting,
$$[K_0, D] + I_2 = D_x$$
where $I_2$ is given as in (3.7). Since $D$ is generic and diagonal, this relationship gives a 1-to-1 correspondence between the independent entries of $K_0$ and those of $I_2$. Following Theorem 2, $D$ and $I_2$ generate all differential invariants and, therefore, so do the entries of $K_0$ and $K_1$, as it was stated in the theorem.

4. Hamiltonian structures in the space of moving coframes

In this section we will define the geometric Hamiltonian Poisson brackets on the space of loops in matrices of the form (3.8). In fact, if we have a matrix of the form (3.8) with periodic entries [M1] showed that, generically, there exists a curve of Lagrangian planes such that $K$ is its associated moving coframe in the sense of the previous section. We refer the reader to [M1] for further information. We now proceed with the construction of the geometric Poisson brackets described in the previous section.

Let $N_0 \subset G_0$ be the isotropy subgroup of
$$\Lambda = \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix}$$
within the subgroup $G_0$. Its Lie algebra is given by

\begin{equation}
\mathfrak{n}_0 = \{ \begin{pmatrix} R_0 & 0 \\ 0 & R_0 \end{pmatrix}, R_0 \text{ skew symmetric} \}. \tag{4.1}
\end{equation}

We call $N = N_0 \cdot G_1$, so its Lie algebra is given by

\begin{equation}
\mathfrak{n} = \{ \begin{pmatrix} R_0 & 0 \\ R & R_0 \end{pmatrix}, R_0 \text{ skew symmetric}, R \text{ symmetric} \}. \tag{4.2}
\end{equation}

Choose $\mathcal{M} \subset \mathcal{L}g^*$ to be given by matrices of the form

\begin{equation}
\begin{pmatrix} A & \alpha I \\ B & -A^T \end{pmatrix}, \tag{4.3}
\end{equation}

where $A(x) \in \mathfrak{gl}(n, \mathbb{R})$, $B(x)$ is symmetric and $\alpha(x) > 0$. Theorem 6 in [M1] states that there exists an open set of $\mathcal{M}$, call it $U$, such that $U/\mathcal{L}N$ can be identified with a section given by all moving coframes of the form (3.8) associated to curves of Lagrangian planes. The quotient is taken under the Kac-Moody coadjoint action of $\mathcal{L}N$ on $\mathcal{L}g^*$ given in (2.5).

The description of the reduced bracket is as follows: Let $h, g : U/\mathcal{L}N \to \mathbb{R}$ be two functionals and let $\mathcal{H}, \mathcal{G} : \mathcal{M} \to \mathbb{R}$ be any two extensions that are constant on the leaves of $\mathcal{L}N$. We define the reduction of (2.4) and (2.6) by the relations

\begin{equation}
\{h, g\}_R(k) = \{\mathcal{H}, \mathcal{G}\}(K), \quad \{h, g\}_0(k) = \{\mathcal{H}, \mathcal{G}\}_0(k) \tag{4.4}
\end{equation}
respectively, where \( k \) represents a point in \( K \) (or the independent entries in \( K \)). For an extension \( H \) to be constant on the leaves of \( LN \) we need the condition

\[
(4.5) \quad \left( \frac{\delta H}{\delta L}(K) \right)_x + \left[ K, \frac{\delta H}{\delta L}(K) \right] \in \mathfrak{n}^0,
\]

to be satisfied for any \( K \in K \), where \( \mathfrak{n} \) is given as in (4.2) and \( \mathfrak{n}^0 \) is its annihilator. We will see that this condition determines enough components of \( \frac{\delta H}{\delta L}(K) \) in terms of \( \frac{\delta h}{\delta K} \) and \( K \) to write all reduced brackets explicitly. Indeed, assume

\[
(4.6) \quad \frac{\delta H}{\delta L}(K) = \begin{pmatrix} H_0 & H_{-1}^+ \\ H_1 & -H_0^T \end{pmatrix}
\]

with \( H_1 \) and \( H_{-1} \) symmetric, \( H_0 \in \mathfrak{gl}(n, \mathbb{R}) \). Let’s denote by \( H_i^+ \) and \( H_i^- \) the symmetric and skew symmetric components of \( H_i \) so that \( H_i = H_i^+ + H_i^- \), and let \( H_0^0 \) represent the diagonal of \( H_0 \). Then (4.5) implies

\[
(4.7) \quad H_0^+ = \frac{1}{2} ((H_{-1})_x + [K_0, H_{-1}])
\]

and

\[
(4.8) \quad (H_{-1}K_1)^- = (H_0^-)_x + [K_0, H_0^-].
\]

With these two conditions, the Poisson reduction of (2.4) can be written as

\[
\{h \cdot g\}^R(k) = \{\mathcal{H}, \mathcal{G}\}(K)
\]

\[
= \left( (H_0^+)_x + [K_0, H_0^+] - (H_{-1}K_1)^+ \begin{pmatrix} 0 & G_{-1}^- \\ G^T_1 & -G_0^T \end{pmatrix} \right)
\]

\[
= 2\langle H_1, G_0^+ \rangle + \langle (H_1)_x + [K_0, H_1], G_{-1} \rangle + 2\langle (H_0^+)_x + [K_0, H_0^+] - (H_{-1}K_1)^+, G_0^+ \rangle + 2\langle (K_1H_0)^+, G_{-1} \rangle,
\]

where \( \langle , \rangle \) is the integral over \( S^1 \) of half the trace. Notice that

\[
2\langle H_1, G_0^+ \rangle - \langle (H_1)_x + [K_0, H_1], G_{-1} \rangle = 0
\]

from (4.7). This is expected since \( H_1 \) cannot be determined from conditions (4.7) and (4.8). Therefore, the reduced bracket becomes

\[
(4.9) \quad \{h \cdot g\}^R(k) = 2\langle (H_0^+)_x + [K_0, H_0^+] - (H_{-1}K_1)^+, G_0^+ \rangle + 2\langle (K_1H_0)^+, G_{-1} \rangle.
\]

Notice also that, since \( \mathcal{H} \) and \( \mathcal{G} \) are extensions of \( h \) and \( g \), which are defined on \( K \), \( H_0^- \) and \( G_0^- \) must coincide with the variational derivative of \( h \) and \( g \), respectively, in the \( K_0 \)-direction, (recall that \( K_0 \) in (3.8) is skew-symmetric). Likewise \( H_0^0 \) and \( G_0^0 \) must coincide with the variational derivative of \( h \) and \( g \) in the \( K_1 \) direction, since \( K_1 \) is diagonal. But, precisely because \( K_1 \) is diagonal, equation (4.8) determines the off-diagonal entries of \( H_{-1} \) in terms of \( H_0^- \), so that \( H_{-1} \) can be written in terms of \( \frac{\delta h}{\delta K} \). Equation (4.7) determines \( H_0^0 \) in terms of \( H_{-1} \) and so \( \{h \cdot g\}^R \) can be found algebraically in terms of \( \frac{\delta h}{\delta K} \) and \( \frac{\delta g}{\delta K} \) using equations (4.7) and (4.8).

The second reduction is simpler. If \( S \) is a constant symmetric matrix then

\[
(4.10) \quad \{h \cdot g\}^R(k) = \left[ \begin{pmatrix} H_0 & H_{-1}^+ \\ H_1 & -H_0^T \end{pmatrix}, \begin{pmatrix} G_0 & G_{-1}^T \\ G_1 & -G_0^T \end{pmatrix} \right]
\]
which also can be explicitly written in terms of \( \frac{\delta h}{\delta k} \) and \( \frac{\delta g}{\delta k} \) and the entries of \( K \).

**Theorem 10.** The reduction of (2.4) and (2.6) to \( K \) results in two Poisson brackets defined by (4.9) and (4.10), respectively, where \( H_0, G_0, H_{-1}, G_{-1} \) are determined uniquely by (4.7) and (4.8) in terms of \( \frac{\delta h}{\delta k}, \frac{\delta g}{\delta k} \) and \( k \).

### 4.1. Geometric evolutions with Hamiltonian invariantization.

If 
\[
\frac{u_t}{u_t} = R(u, u_1, u_2, \ldots)
\]
is a geometric evolution of Lagrangian planes (that is, invariant under the action of the symplectic group), then it is known that the evolution is given by an evolution of the form (2.7) where \( F \) has in columns a classical moving frame and \( r \) is a vector of differential invariants, that is, a vector whose entries depend on the independent entries of \( K \) as in (3.8) and their derivatives with respect to \( x \).

One can write a geometric evolution as an evolution in \( g_{-1} \) using the left invariant identification of the tangent to \( G/H \) with \( g_{-1} \). Following the description in [M1], the evolution is given by
\[
\rho_{-1}^{-1}(\rho_{-1})_x = \text{Ad}(\rho_0)v_r
\]
where \( v_r \) is an (differential) invariant element of \( g_{-1} \) (that is, depending on \( k \) and its derivatives) defined by \( r \) under the identification of the tangent to the manifold with \( g_{-1} \). In our particular case such an identification would be written as
\[
\begin{pmatrix}
I & -u \\
0 & I
\end{pmatrix}
\begin{pmatrix}
u_r \\
0
\end{pmatrix}
= \begin{pmatrix}
0 & \Theta^T \\
\Theta & 0
\end{pmatrix}
\begin{pmatrix}
u_r \\
0
\end{pmatrix}
= \begin{pmatrix}
0 & \Theta u^{-\frac{1}{2}} \\
\Theta u^{-\frac{1}{2}} & 0
\end{pmatrix}
\begin{pmatrix}
u_r \\
0
\end{pmatrix}
\]
so that any geometric evolution of the form (2.7) is given by
\[
u_t = u_{1}^{\frac{1}{2}} \Theta^T v_r \Theta u_{1}^{\frac{1}{2}}
\]
for some choice of symmetric matrix \( v_r \) whose entries depend on \( k \) and their derivatives. The following theorem is now straightforward.

**Theorem 11.** A classical moving frame for a curve of Lagrangian planes under the action of the symplectic group is given by
\[
u_t = u_{1}^{\frac{1}{2}} \Theta^T E_{i,j} \Theta u_{1}^{\frac{1}{2}}
\]
where \( E_{i,j} \) has a 1 in the \( (i,j) \) entry and zeroes elsewhere, and where \( i \leq j \). Furthermore, any evolution of curves of Lagrangian planes of the form \( u_t = F(u, u_1, u_2, \ldots) \) which is invariant under the action of the symplectic group can be written as (4.11) for some choice of differential invariant symmetric matrix \( v_r \).

Let’s go back to Hamiltonian brackets. Following Theorem 6, if \( u_t \) evolves following (4.11) and \( h : K \to \mathbb{R} \) is such that one can find an extension \( H \) constant on the leaves of \( \mathcal{LN} \) with \( H_{-1} = v_r \), then we can conclude that the invariantization of the geometric flow (2.7) is Hamiltonian with respect to the first reduce bracket and its associated Hamiltonian is \( h \).

If we evaluate the structure equation for the Maurer-Cartan form of \( Sp(2n) \) on the pair \( (\rho_x, \rho_t) \) along the 1-parameter family of curves \( \rho \), and assuming \( x \) and \( t \) are independent, one obtains the equation
\[
K_t = N_x + [K, N]
\]
where $N = \rho^{-1}\rho_t$. If $u$ evolves following (4.11) with $\Theta$ given by the choice of $\rho$, then it is a straightforward calculation to check that $N_{-1} = v_r$.

**Theorem 12.** Equation (4.12) is the invariantization of (4.11). Furthermore, equation (4.12) and $N_{-1} = v_r$ completely determine $N$ algebraically.

**Proof.** We only need to show that (4.12) and $N_{-1}$ determine $N$. Indeed, equation (4.12) can be written as

\[
\begin{pmatrix}
K_0 & I \\
K_1 & K_0
\end{pmatrix}_t = \begin{pmatrix} N_0 & v_r \\
N_1 & -N_0^T
\end{pmatrix} + \left[ \begin{pmatrix} K_0 & I \\
K_1 & K_0
\end{pmatrix}, \begin{pmatrix} N_0 & v_r \\
N_1 & -N_0^T
\end{pmatrix} \right]
\]

\[
= \begin{pmatrix} (N_0)_x + N_1 + [K_0, N_0] - v_r K_1 & (v_r)_x - N_0^T + [K_0, v_r] - N_0 \\
(N_1)_x + [K_0, N_1] + K_1 N_0 + N_0^T K_1
\end{pmatrix}.
\]

The matrix $N$ does not need to be computed from $\rho$. In fact, since $N$ is a solution of (4.13) we need $N_0$ and $N_1$ to satisfy

\[
0 = (N_0^+)_x + N_1 + [K_0, N_0^+] - (v_r K_1)^+ \\
2 N_0^+ = (v_r)_x + [K_0, v_r] \\
K_1 N_0 + N_0^T K_1 + (N_1)_x + [K_0, N_1]
\]

diagonal.

Equations (4.15) and (4.16) determine $N_0$ completely in terms of $v_r$ and $K$, while equation (4.14) determines $N_1$.

**4.2. Evolutions of Lagrangian planes whose invariantization is a system of decoupled KdV equations.** In this section we will show that the submanifold

\[
K_1 = \{ K = \begin{pmatrix} 0 & I \\
K_1 & 0
\end{pmatrix} \} \subset \mathcal{K}
\]

is a Poisson submanifold of $\mathcal{K}$ and that the Poisson structures induced by $\{,\}_0^R$ and $\{,\}_1^R$ are a family of $n$ decoupled first and second KdV Hamiltonian structures. We will also find the geometric evolutions of Lagrangian planes whose invariantization is a system of decoupled KdV equations, assuming the initial conditions are taken among curves whose invariants belong to $K_1$. When the complete manifold $\mathcal{K}$ is considered we will illustrate the situation by showing that for $G = Sp(4)$ two independent Nijenhuis operators can be found using $\{,\}_1^R$ and two invertible Poisson structures found among $\{,\}_0^R$ for two different choices of $S$. Although it is not clear of these operators are linked to any completely integrable system, this example shows the simplicity of the calculations involved in finding geometric Poisson brackets in particular cases.

**Theorem 13.** The submanifold $K_1 \subset \mathcal{K}$ is a Poisson submanifold and the brackets $\{,\}_1^R$ and $\{,\}_0^R$ restricted to $K_1$ are equal to a system of decoupled second and first KdV Hamiltonian structures, respectively.

**Proof.** One can easily see that $\{,\}_1^R$ and $\{,\}_0^R$ can be restricted to $K_1$ and find the restriction (for example, if $k_1$ are the coordinates in the $K_0$-direction, it is trivial to check that $\{h, k_1\}^R_0(k) = \{h, k_1\}^R_0(k) = 0$ if $k \in K_1$). Indeed, if $h : K_1 \subset \mathcal{K} \to \mathbb{R}$, an extension of $h$ to $\mathcal{M}$ can be assumed to have vanishing $H_0^-$ in (4.6), since it is determined by the variational derivative of $h$ in the $K_0$-direction. Therefore (4.8) becomes

\[
(H_{-1} K_1)^- = 0.
\]
Since $K_1$ is diagonal, we have that for generic $K_1$, $H_{-1}$ has zero entries outside the diagonal. With this information (4.7) is given by

$$H_0 = H_0^+ = \frac{1}{2}(H_{-1})_x + \frac{1}{2}[K_0, H_{-1}].$$

From here, if $h$ and $g$ are two such functionals we can choose such extensions and (4.9), for $K \in K_1$, would be given by

$$\{h, g\}^R(k) = 2\langle (H_0^+)_x - H_{-1}K_1, G_0^+ \rangle + 2\langle K_1H_0, G_{-1} \rangle$$

where $H_0$ is now diagonal also since $K_0 = 0$. Since all matrices involved are diagonal and $H_{-1}$ and $G_{-1}$ contain in their diagonals the variational derivatives of $h$ and $g$ in the direction of $K_1$, this is a system of decoupled second Hamiltonian structures for KdV.

Our second bracket is even simpler. Expression (4.10) becomes

$$\{h, g\}_0^R(k) = \langle G_{-1}, S(H_{-1})x + (H_{-1})_xS \rangle = 2\langle G_{-1}, S(H_{-1})x \rangle$$

since only the diagonal entries of $S$ are relevant. This is a family of decoupled first KdV structures. Indeed, $D^3 + 2k_iD + (k_i)_x$ and $D$ are the standard Hamiltonian structures used to integrate the KdV equation. ♣

Next, we want to study which geometric evolutions of Lagrangian planes will have Hamiltonian invariantizations with respect to the decoupled system of KdV structures. If a geometric evolution is of the form (4.11) for some invariant symmetric matrix $v_r$, the condition to have a Hamiltonian invariantization with respect to $\{\cdot, \cdot\}^R$ was given in Theorem 6. Indeed, if there is $h : K \to \mathbb{R}$ such that an extension $\mathcal{H}$ constant on the leaves of $LN$ holds $H_{-1} = v_r$, then its invariantization will be Hamiltonian with respect to $\{\cdot, \cdot\}^R$ and its Hamiltonian functional will be $h$. In particular, if we restrict ourselves to $K_1$, we saw above that $H_{-1}$ can be chosen to be a diagonal matrix with diagonal given by $\frac{\delta h}{\delta k_i}$. The following theorem has been proved by now.

**Theorem 14.** Let $u(t, x)$ be a family of curves of Lagrangian planes evolving following the geometric evolution

$$u_t = u_1^4 \Theta^T \begin{bmatrix} \delta h \\ \delta k \end{bmatrix} \Theta u_1^4$$

for some $h : K_1 \to \mathbb{R}$, where $\begin{bmatrix} \delta h \\ \delta k \end{bmatrix}$ is the diagonal matrix with $\frac{\delta h}{\delta k_i}$ down its diagonal. Then, its invariantization is

$$(K_1)_t = (D^3 + K_1D + (K_1)_x) \begin{bmatrix} \delta h \\ \delta k \end{bmatrix}$$

$$(K_0)_t = 0$$

if we choose initial conditions $u(0) = u_0$ for which $K_0 = 0$. In particular, if $\begin{bmatrix} \delta h \\ \delta k \end{bmatrix} = K_1$, then the geometric evolution is integrable since its invariantization is
a system of decoupled KdV equations. In that case, the geometric evolution is given by

\begin{equation}
(4.18) \quad u_t = u_1^\frac{1}{2} S(u) u_1^\frac{1}{2} = u_3 - \frac{3}{2} u_2 u_1^{-1} u_2.
\end{equation}

This theorem is a generalization of a well-known fact for 1-dimensional projective curves, namely that the Schwarzian KdV equation

\begin{equation}
(4.19) \quad u_t = u_1 S(u) = u_3 - \frac{3}{2} \frac{u_2^2}{u_1}
\end{equation}

induces KdV on \( S(u) \). Hence, we call (4.18) the Lagrangian Schwarzian KdV equation.

It is important to notice that a general evolution of the form (4.11) does not necessarily preserve \( K_1 \). In fact, from (4.13), we see that \( K_0 \) will be preserved whenever

\[ N_1 = -(N_0)_x - [K_0, N_0] + \nu_r K_1, \]

that is, we need

\[ (N_0^+)_x + [K_0, N_0^-] = (\nu_r K_1)^- \]

to be satisfied. Since \( N_0 \) is written in terms of \( K \) and \( \nu_r \), the above is a condition on \( \nu_r \) and those evolutions who hold it will preserve the value of \( K_0 \) given by the initial condition. In particular, if \( K_0 = 0 \) the \( K_0 \) preserving condition we obtain is

\[ (N_0^-)_x = (\nu_r K_1)^- \]

where \( N_0^+ = \frac{1}{2} (\nu_r)_x \) and \( N_1 = (\nu_r K_1)^+ - \frac{1}{2} (\nu_r)_{xx} \), and where the off diagonal entries of \( K_1 N_0 + N_0^T K_1 \) and \(-N_1\) must coincide.

**Corollary 1.** If \( \nu_r \) is a diagonal matrix, evolution (4.11) preserves the submanifold \( K_1 \).

This is now trivial since in this case both \( N_0^+ \) and \( N_1 \) are diagonal and hence the conditions above are satisfied.

Finally, let \( V = -\frac{1}{2} \Theta u_1^{-1/2} u_2 u_1^{-1/2} \Theta^T \). We will show that, on \( K_1 \), if \( u \) is a solution of (4.11) with \( \nu_r = K_1 \), then \( V \) satisfies the modified KdV equation. In fact, it is trivial to check that

\[ K_1 = V_x + V^2. \]

This transformation was called the generalized Miura transformation (or Lagrangian Miura transformation) in [Ov].

**Theorem 15.** Assume \( u \) evolves following (4.11) and assume that \( \nu_r = K_1 \) and hence \( K_1 \) is preserved. Then \( V \) evolves following a modified matrix KdV equation.

Notice that \( V \) is not necessarily diagonal.

**Proof.** Let \( N_0 \) be the \( g_0 \) component of \( \rho^{-1} \rho_t \) and \( K_0 \) the one for \( \rho^{-1} \rho_x \). It is simple to see that

\[ N_0 = -g_x g^{-1} + \nu_r V, \quad K_0 = -g_x g^{-1} + V \]

where \( g = \Theta u_1^{-1/2} \). Differentiating the first equation with respect to \( x \) and the second with respect to \( t \) and using \( K_0 = 0 \) we get the evolution of \( V \) to be

\begin{equation}
(4.19) \quad V_t = -(N_0)_x + [\nu_r V - N_0, V] + (v_r V)_x.
\end{equation}
Now, equations (4.14, 4.15, 4.16) tell us that, on $K_1, N_0$ and $N_1$ are diagonal whenever $v_r$ is, and, furthermore, $N_0^+ = N_0 = \frac{1}{2}(v_r)_x$. If we substitute

$$v_r = K_1 = V_x + V^2$$

in (4.19) we obtain that the evolution of $V$ is given by

$$V_t = -\frac{1}{2}V_{xxx} + \frac{3}{2}V_x V^2 + \frac{3}{2}V^2 V_x$$

which is the matrix modified KdV equation.

\[\Box\]

4.3. \textbf{Nijenhuis operators in the case $G = Sp(4)$}. Let's consider the simpler case of $G = Sp(4)$. In this case we can obtain two Nijenhuis operators from two invertible Hamiltonian structures. Of course, these operators can exist independently from any completely integrable system and it is not clear that these exist. Still, this example show the simplicity of our calculations. Notice that $sp(2)$ is equivalent to $sl(2)$ which is the very well known case of 1-dimensional projective geometry. In the $sl(2)$ case one obtains the well known second and first Hamiltonian structures for KdV as reduced brackets and KdV itself as associated Hamiltonian evolution.

Let’s find explicitly the reduction of (2.6), since it is simpler, and we will attack the first reduction afterwards. In this particular case $K$ is defined by matrices of the form

$$K = \begin{pmatrix}
0 & -k_1 & 1 & 0 \\
k_1 & 0 & 0 & 1 \\
k_2 & 0 & 0 & -k_1 \\
0 & k_3 & k_1 & 0
\end{pmatrix}.$$

Assume that $h, g : K \to \mathbb{R}$ are two functionals on $K$ and let’s denote by $h_i$ the variational derivative of $h$ in the $k_i$ direction. Likewise with $g$. In that case, if $\mathcal{H}$ is an extension of $h$ its variational derivative will be given by a matrix of the form

$$\begin{pmatrix}
\alpha & h_1 + \eta & h_2 & \beta \\
-h_1 + \eta & \gamma & \beta & h_3 \\
* & * & \alpha & h_1 + \eta \\
* & * & -h_1 + \eta & \gamma
\end{pmatrix}$$

where the asterisks indicate entries that will not be involved in the definition of the bracket. Equations (4.7) and (4.8) become, in this case

$$2 \begin{pmatrix}
\alpha & \eta & \gamma
\end{pmatrix} = \begin{pmatrix}
h_2 & \beta & h_3
\end{pmatrix}_x + \left[ \begin{pmatrix}
0 & -k_1 \\
k_1 & 0
\end{pmatrix}, \begin{pmatrix}
h_2 & \beta \\
h_3 & h_1 + \eta
\end{pmatrix}_x \right].$$

and

$$\frac{1}{2} \begin{pmatrix}
\beta (k_1 - k_2) & 0 \\
0 & \beta (k_2 - k_3)
\end{pmatrix} = \begin{pmatrix}
0 & h_1 \\
-h_1 & 0
\end{pmatrix}_x + \left[ \begin{pmatrix}
0 & -k_1 \\
k_1 & 0
\end{pmatrix}, \begin{pmatrix}
0 & h_1 \\
-h_1 & 0
\end{pmatrix}_x \right].$$

These two equations determine the values

$$\beta = \frac{-2}{k_2 - k_3} (h_1)_x, \quad \alpha = \frac{1}{2} (h_2)_x + \frac{k_1}{k_2 - k_3} (h_1)_x,$$

$$\gamma = \frac{1}{2} (h_3)_x - \frac{k_1}{k_2 - k_3} (h_1)_x, \quad \eta = \left( -\frac{1}{k_2 - k_3} (h_1)_x \right)_x + \frac{1}{2} k_1 (h_2 - h_3).$$

Assume a constant symmetric matrix $S$ is given by

$$S = \begin{pmatrix}
a & b \\
b & c
\end{pmatrix},$$
then the reduced Poisson bracket (4.10) is written as
\[
\{ h, g \}_R^R(k) = a [g_2 \alpha_h + \beta_g (\eta_h + h_1) - \alpha_g h_2 - (g_1 + \eta_g) \beta_h] + b [g_2 (\eta_h - h_1) + \beta_g \gamma_h + \beta_g \alpha_h + g_3 (\eta_h + h_1) - \alpha_g \beta_h - (g_1 + \eta_g) h_3 - (\eta_g - g_1) h_2 - \gamma_g \beta_h] + c [g_3 \gamma_h + \beta_g (\eta_h - h_1) - (\eta_g - g_1) \beta_h - \gamma_g h_3].
\]
Since the reduction produces a Poisson bracket for any choice of \( S \) above we have three generating Poisson brackets defined by the coefficients of \( a, b \) and \( c \). In fact, it is given by
\[
\text{Definition 4.} \quad \text{A Poisson bracket is said to be invertible or symplectic if, when written as in (4.22), any matrix of differential operators } D \text{ such that } D_0 D = 0 \text{ necessarily vanishes.}
\]
With this definition it is trivial to check that both \( D_0^a \) and \( D_0^c \) are invertible. If one finds out the bracket defined by the coefficient of \( b \), the bracket will not be invertible. In fact, it is given by
\[
D_0^b = \begin{pmatrix}
0 & 1 + 2D \frac{k_1}{k_2 - k_3} D & -1 + 2D \frac{k_1}{k_2 - k_3} D \\
-1 - 2D \frac{k_1}{k_2 - k_3} D & 0 & -k_1 \\
1 - 2D \frac{k_1}{k_2 - k_3} D & k_1 & 0
\end{pmatrix}.
\]
Having invertible Poisson structures allows us to generate Nijenhuis operators. But first we will find an expression for the reduction of (2.4). Again, we substitute all known values into (4.9) and, after long but trivial calculations we get that the matrix of differential operators defining \( \{ , \}^R(k) \) is given by \( D = (D_{ij}) \) where
\[ D_{11} = 2D \frac{D^3}{k_2 - k_3} - \frac{1}{k_2 - k_3} D - 2D \frac{1}{k_2 - k_3} D - 2D \frac{k_1}{k_2 - k_3} D - \frac{k_1}{k_2 - k_3} D - 4D \frac{\frac{1}{2} k_2}{k_2 - k_3} D - \frac{1}{k_2 - k_3} D \\
\quad - 4D \frac{1}{k_2 - k_3} D - \frac{k_1^2}{k_2 - k_3} D - 2D \frac{1}{k_2 - k_3} D - 2D \frac{k_1 + k_3}{k_2 - k_3} D - \frac{k_2 + k_3}{k_2 - k_3} D - \frac{1}{k_2 - k_3} D + 2D \\
\quad D_{22} = \frac{1}{4} D^3 + \frac{1}{4} k_1 Dk_1 + \frac{1}{2} Dk_1^2 + \frac{1}{2} k_1^2 D + \frac{1}{2} Dk_2 + \frac{1}{2} k_2 D \\
\quad D_{31} = -\frac{1}{4} D^3 + \frac{1}{2} k_1 Dk_1 + \frac{1}{2} Dk_1^2 + \frac{1}{2} k_1^2 D + \frac{1}{2} Dk_3 + \frac{1}{2} k_3 D \\
\quad D_{21} = -\frac{1}{2} D^2 \frac{k_1}{k_2 - k_3} D - Dk_1 D + \frac{1}{k_2 - k_3} D - \frac{k_1 + k_3}{k_2 - k_3} D \\
\quad \quad + \frac{k_1 (k_2 + k_3)}{k_2 - k_3} D + \frac{k_1 k_2}{k_2 - k_3} D \\
\quad D_{31} = \frac{1}{2} D^2 \frac{k_1}{k_2 - k_3} D + Dk_1 D + \frac{1}{k_2 - k_3} D + k_1 D^2 - \frac{1}{2} Dk_3 - \frac{1}{2} k_3 D \\
\quad \quad - \frac{k_1 (k_2 + k_3)}{k_2 - k_3} D - \frac{k_1 k_2}{k_2 - k_3} D \\
\quad D_{32} = -\frac{1}{2} k_1 Dk_1 - \frac{1}{2} k_1^2 D - \frac{1}{2} Dk_2^2 \\
\]

**Theorem 16.** The two tensors \( N_1 = D(D_0^a)^{-1} \) and \( N_2 = D(D_0^b)^{-1} \) are two Nijenhuis operators defined on \( K \).

Of course, if \( k_1 = 0 \) and we consider only functionals \( h, g \) defined on \( K_1 \) (that is \( \frac{\delta h}{\delta k_1} = \frac{\delta g}{\delta k_1} = 0 \)) then \( D_0^a + D_0^b \) and \( D \) are the standard decoupled first and second Hamiltonian structures for KdV.

**References**


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