Conformal geometry of hyperbolic surfaces in the Lagrangian-Grassmannian and 2nd order PDE

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Outline

1. From 2nd order PDE to surfaces in $LG(2, 4)$
2. Geometry of $LG(2, 4)$: conformal structure, “spheres”, and moving frames
3. $CSp_4(\mathbb{R})$ classification of hyperbolic (timelike) surfaces / (non-linear) hyperbolic PDE
2nd order PDE and Jet Space

A PDE $F(x, y, z, z_x, z_y, z_{xx}, z_{xy}, z_{yy}) = 0$ is a 7-dim hypersurface $\Sigma$ in $J^2 = J^2(\mathbb{R}^2, \mathbb{R}) : (x, y, z, p, q, r, s, t)$. $J^2$ has contact system $\mathcal{C}^2$:

$$\sigma = dz - p dx - q dy, \quad \sigma^1 = dp - r dx - s dy, \quad \sigma^2 = dq - s dx - t dy$$

Assume: $\Sigma \cap \{\text{any fibre } J^2 \to J^1\}$ is a 2-dim surface.

**Definition**

$\Sigma, \tilde{\Sigma}$ are (locally) contact-equiv. if $\exists \phi : J^2 \to J^2$ (local) contact transf. s.t. $\phi(\Sigma) = \tilde{\Sigma}$. A contact symmetry is a self-equivalence.

General goal: Find invariants distinguishing local equiv. classes.

**KEY FACT:** The fibres of $J^2 \to J^1$ are diffeo. to $LG(2, 4)$. 
Sp$_4$(R) and the Lagrangian–Grassmannian

- Std. symplectic form $\eta$ on $\mathbb{R}^4$ with matrix $J = \begin{pmatrix} 0 & l_2 \\ -l_2 & 0 \end{pmatrix}$.
- $LG(2, 4) = \{\eta\text{-isotropic 2-planes in } \mathbb{R}^4\}$, e.g. $o = \mathbb{R}\{e_1, e_2\}$.
- $Sp_4(\mathbb{R}) = \{X : X^T J X = J\}$ acts transitively on $LG(2, 4)$, so $LG(2, 4) = Sp_4(\mathbb{R})/P_2$, where $P_2 = \text{stab. of } o$.
- $CSp_4(\mathbb{R}) = \{X : X^T J X = \rho J\}$ also acts trans. on $LG(2, 4)$. 
PDE $\leadsto$ surfaces in $LG(2,4)$

**Theorem (Backlund Thm)**

*Any contact transf. of $J^2$ is the prolongation of one on $J^1$.***

$(J^1, \sigma)$ is a contact mfld; $\eta = d\sigma$ restricted to $C = \{\sigma = 0\}$ is a symplectic form. Let $\phi : J^1 \rightarrow J^1$ be contact, so $\phi^* \sigma = \lambda \sigma$. Then

$$\phi^* \eta = \phi^* (d\sigma) = d(\phi^* \sigma) = d\lambda \wedge \sigma + \lambda \eta,$$

so $\phi_*(C_\xi, \eta_\xi) \rightarrow (C_{\xi'}, \eta_{\xi'})$ is a conformal symplectomorphism. If $\phi$ fixes $\xi \in J^1$, it acts on $J^2|_\xi := LG(C_\xi, \eta_\xi)$ by $CSp(C_\xi, \eta_\xi)$. Since a PDE $\Sigma$ intersects $J^2|_\xi$ as a 2-dim surface, we are led to:

**Study the local geometry of surfaces in $LG(2,4)$ mod $CSp_4(\mathbb{R})$**
PDE with constant symplectic invariants

Mod basept, $LG(C_\xi, \eta_\xi) \cong LG(2, 4)$ is canonical. Thus, a $CSp_4(\mathbb{R})$-inv. $\kappa$ of surfaces in $LG(2, 4)$ induces a $CSp(C_\xi, \eta_\xi)$-inv. $\kappa_\xi$ on each fibre $J^2|_{\xi}$. Define $\tilde{\kappa}$ on $\Sigma$ by $\tilde{\kappa}(\tilde{\xi}) = \kappa_\xi(\tilde{\xi})$. Under a contact transformation $\phi : J^1 \rightarrow J^1$ s.t. $\phi(\Sigma) = \Sigma'$,

$$\tilde{\kappa} : \Sigma \rightarrow \mathbb{R} \quad \mapsto \quad \tilde{\kappa}' = \tilde{\kappa} \circ \text{pr}(\phi^{-1}) : \Sigma' \rightarrow \mathbb{R}.$$ 

Thus, $\tilde{\kappa}$ may not be contact-invariant. However:

**Lemma (T. 2010)**

*All PDE satisfying $\tilde{\kappa} = c = \text{const}$ form a contact-invariant class.*

**Example**

The elliptic / parabolic / hyperbolic (contact-inv.) classification comes from a discrete $CSp_4(\mathbb{R})$-invariant of surfaces in $LG(2, 4)$. 

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Dennis The  Surfaces in LG(2,4) and 2nd order PDE
By Plücker: $Gr(2,4) \hookrightarrow \mathbb{P}(\Lambda^2 \mathbb{R}^4)$, and $LG(2,4) \hookrightarrow \mathbb{P}V = \mathbb{RP}^4$, where $V = \Lambda^2_0 \mathbb{R}^4 = \{z \in \Lambda^2 \mathbb{R}^4 : \eta(z) = 0\}$. $V$ admits 

$$\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}, \quad (z_1, z_2) \to (\eta \wedge \eta)(z_1 \wedge z_2),$$

which has sig. $(2,3)$. $LG(2,4)$ is identified with the space of null lines in $V$, which is a quadric hypersurface in $\mathbb{P}V$:

$$LG(2,4) = Q = \{[z] \in \mathbb{P}V : \langle z, z \rangle = 0\}.$$

This realizes $Sp_4(\mathbb{R}) \cong Spin(2,3)$. $\langle \cdot, \cdot \rangle$ induces a Lorentzian conf. str. on $Q = LG(2,4)$, i.e. a canonical cone field.
Elliptic / parabolic / hyperbolic

Lorentzian conformal structure: \([\mu] = [drdt - ds^2]\).
If \(F(r, s, t) = 0\) is \(M\), then \(n = F_t \partial_r - \frac{1}{2} F_s \partial_s + F_r \partial_t\) spans \(NM\)
\(\Rightarrow \mu(n, n) = F_r F_t - \frac{1}{4} F_s^2\) has intrinsic sign. (+:ell, 0:par, -:hyp)
Spheres

Definition

For any \([z] \in \mathbb{P}V\), we refer to \(S[z] = \mathbb{P}(z^\perp) \cap Q\) as a sphere.

There is a natural basis \(B\) on \(V\), wrt which \(Q\) has local coords \(w = (1, r, s, t, rt - s^2)\). Given \(z = (z_0, \ldots, z_4) \in V\), \(S[z]\) looks like

\[
0 = \langle w, z \rangle = -z_0(rt - s^2) + z_3r - 2z_2s + z_1t - z_4.
\]

This is exactly the Monge–Ampère eqn! Fibrewise, it’s a sphere.

Theorem

1. \(CSp_4(\mathbb{R})\) acts transitively on \(\mathcal{H}_- = \{[z] : \langle z, z \rangle < 0\}\).
2. The class of (hyperbolic) MA eqns is contact-invariant.

NOTE: Last proof – Gardner–Kamran (1993). This proof is new. Invariance of the parabolic and elliptic MA eqns is similar.
From 2nd order PDE to surfaces in LG(2, 4)

Geometry of LG(2, 4)

CSp₄(ℝ)-classification of hyperbolic surfaces

Moving frames – geometric picture

For hyperbolic (timelike) surfaces $M$, use frames $v = (v_0, v_1, v_2, v_3, v_4)$ satisfying

$$\langle v_i, v_j \rangle = \begin{pmatrix}
0 & 0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & -2 & 0 \\
-1 & 0 & 0 & 0 & 0
\end{pmatrix}$$

Moving frame adaptations:
- 0th order: $[v_0] \in M$
- 1st order: null lines $\overline{v_0v_1}, \overline{v_0v_2}$
- 2nd order: $S_{v_3}$ (central tangent sphere)
- 3rd order: $S_{v_1}, S_{v_2}$ (null cones)
- 3rd order: $[v_4]$ is the unique pt in $S_{[v_1]} \cap S_{[v_2]} \cap S_{[v_3]}$

NOTE: 3rd order normalizations are not possible for spheres (a.k.a. MA eqns).

**Definition**

The conjugate manifold $M'$ of $M$ is the image of $M \to Q$, $p \mapsto [v_4|_p]$. Given a PDE $\Sigma$, can fibrewise construct the conjugate PDE $\Sigma'$.
**Classification of hyperbolic surfaces / PDE (T. 2010)**

**Figure:** Classification of hyperbolic surfaces up to $CSp_4(\mathbb{R})$-equivalence; also, classification of hyperbolic PDE up to contact-equivalence.
Examples

Some examples of hyperbolic surfaces \( M \subset LG(2, 4) \):

- 2-isotropic \([CSp_4(\mathbb{R})\text{-transitive family}]\): All are of the form
  \[ a(rt - s^2) + br + cs + dt + e = 0 \]
- 2-parabolic:
  - \( \dim(M') = 0 \) \([CSp_4(\mathbb{R})\text{-transitive family}]\): \( s = t^2 \)
  - \( \dim(M') = 1 \): \( s = \exp(t) \) or \( s = \ln(t) \).
  - \( F(s, t) = 0 \) or \( F(r, s) = 0 \) non-linear: \( \dim(M') \leq 1 \).
  - \( \dim(M') = 2 \): ?? \([F(r, s, t) = 0 \text{ must have } F_rF_sF_t \neq 0] \)
- 2-generic:
  - \( \dim(M') = 0 \):
    - \( 3rt^3 + 1 = 0 \)
    - \( \frac{(3r - 6st + 2t^3)^2}{(2s - t^2)^3} + c = 0 \) for \( c < 0 \) or \( c > 4 \). [As PDE, all have 9-dim contact sym alg – a parabolic subalg of (non-cpt) \( g_2 \).]
  - \( \dim(M') = 1 \): ??
  - \( \dim(M') = 2 \): \( r = \exp(t) \)

(NOTE: For corresp. PDE, substitute \( r = z_{xx}, s = z_{xy}, t = z_{yy} \).)
Monge–Ampère invariants


**Theorem (T. 2010)**

Let \( M \subset \text{LG}(2,4) \) be hyperbolic, given locally by \( F(r, s, t) = 0 \). Endow \( M \) with any null parametrization \( u, v \). Then the Monge–Ampère invariants are:

\[
I_1 = \det \begin{pmatrix} r_{uu} & s_{uu} & t_{uu} \\ r_u & s_u & t_u \\ r_v & s_v & t_v \end{pmatrix}, \quad I_2 = \det \begin{pmatrix} r_{vv} & s_{vv} & t_{vv} \\ r_u & s_u & t_u \\ r_v & s_v & t_v \end{pmatrix}
\]

or, letting \( n_1 = (r_u, s_u, t_u) \), \( n_2 = (r_v, s_v, t_v) \), we have

\[
I_1 = n_1^T \text{Hess}(F) n_1, \quad I_2 = n_2^T \text{Hess}(F) n_2.
\]

<table>
<thead>
<tr>
<th>Type</th>
<th>( \text{CSp}_4(\mathbb{R}) )-invariant classification</th>
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<tbody>
<tr>
<td>2-isotropic</td>
<td>( I_1 = I_2 = 0 )</td>
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<tr>
<td>2-parabolic</td>
<td>exactly one of ( I_1 ) or ( I_2 ) is zero</td>
</tr>
<tr>
<td>2-generic</td>
<td>2-hyp: ( I_1 I_2 &lt; 0 ); 2-ell: ( I_1 I_2 &gt; 0 )</td>
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Invariants in the generic case

Let $\nu = r_u t_v + r_v t_u - 2s_u s_v$ and $l_3 = \det \begin{pmatrix} r_{uv} & s_{uv} & t_{uv} \\ r_u & s_u & t_u \\ r_v & s_v & t_v \end{pmatrix}$. Invariants are

$$\kappa_1 = \frac{\nu^3}{|l_1| |l_2|} \left( \frac{|l_1|^{3/4} |l_2|^{1/4}}{|\nu|^{3/2}} \right)_v,$$

$$\kappa_2 = \frac{|\nu|^3}{|l_1| |l_2|} \left( \frac{|l_1|^{1/4} |l_2|^{3/4}}{|\nu|^{3/2}} \right)_u,$$

$$\tau = \frac{\text{sgn}(l_1 \nu) \nu^3}{|l_1 l_2|^{1/2}} \left( \frac{2l_3}{\nu^3} - \frac{\nu^2}{l_1 l_2} \left( \frac{l_3}{\nu^2} \right)_{uv} \right).$$

Definition

Call $\kappa := \kappa_1 \kappa_2$ the conformal Gaussian curvature.

Theorem (T. 2010)

1. $\tau$ and $\kappa := \kappa_1 \kappa_2$ are $\text{CSp}_4(\mathbb{R})$-invariants of unparam. surfaces.
2. For "general" generic $M$, $\tau$ is a 3rd order diff. fcn. of $\kappa_1, \kappa_2$.
3. If $M, M'$ are 2-generic, then $(M')' = M$ iff $\tau = 0$. 
Questions

- Corresponding coframe adaptations for PDE?
- Invariants for elliptic and parabolic surfaces in $LG(2, 4)$?
- $CSp_{2n}(\mathbb{R})$-submanifold theory in $LG(n, 2n)$?
- What does the “conjugate” PDE say about the original PDE?