DISCRETE MOVING FRAMES AND APPLICATIONS

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ABSTRACT. Group based moving frames have a wide range of applications, from the classical equivalence problems in differential geometry to more modern applications such as computer vision. Two important applications explored by the authors have been to curvature flows and their relations to integrable systems, and to the calculus of variations. Here we describe what we call a discrete group based moving frame, which is essentially a sequence of moving frames with overlapping domains, and we show that this offers significant computational advantages over a single moving frame for the main application we discuss and develop, namely, to discrete integrable systems. We demonstrate that the discrete analogues of some curvature flows naturally lead to Hamiltonian pairs, which generate integrable differential-difference systems. In particular, we show that in the centro-affine plane and the projective space, the Hamiltonian pairs obtained can be transformed into the known Hamiltonian pairs for the Toda and modified Volterra lattices respectively under Miura transformations. We also describe in detail the case of discrete flows in the homogeneous 2-sphere and we obtain realizations of equations of Volterra type as evolutions of polygons on the sphere.

1. INTRODUCTION

The notion of a moving frame is associated with Élie Cartan [2], who used it to solve equivalence problems in differential geometry, relativity, and so on. Moving frames were further developed and applied in a substantial body of work, in particular to differential geometry and (exterior) differential systems, see for example papers by Green[12] and Griffiths[13]. From the point of view of symbolic computation, a breakthrough in the understanding of Cartan’s methods came in a series of papers by Fels and Olver[6, 7], Olver[37, 38], Hubert[16, 17, 18], and Hubert and Kogan[19, 20], which provide a coherent, rigorous and constructive moving frame method free from any particular application, and hence applicable to a huge range of examples, from classical invariant theory to numerical schemes.

For the study of differential invariants, one of the main results of the Fels and Olver papers is the derivation of symbolic formulae for differential invariants and their invariant differentiation. The book [30] contains a detailed exposition of the calculations for the resulting symbolic invariant calculus. Applications include the integration of Lie group invariant differential equations, to the Calculus of Variations and Noether’s Theorem, (see also [26], [11]), and to integrable systems ([32], [33], [34], [35]).

In this paper we are interested in applications of moving frames to discrete problems, with the first results for the computation of invariants using group-based

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moving frames given by Olver [37]. Modern applications to date include computer vision [38] and numerical schemes for systems with a Lie symmetry [3, 23, 24, 25, 31].

While moving frames for discrete applications as formulated by Olver do give generating sets of discrete invariants, the recursion formulae for differential invariants that were so successful for the application of moving frames to calculus based applications do not generalize well to these discrete invariants. In particular, these generators do not seem to have recursion formulae under the shift operator that are computationally useful.

To overcome this computational problem, we introduce a discrete moving frame which is essentially a sequence of frames\(^1\), and prove discrete recursion formulae for a small computable generating sets of invariants, which we call the discrete Maurer-Cartan invariants. We show that our definitions and constructions arise naturally and are useful for our main application of interest, which is to discrete integrable systems arising as invariantizations of polygon evolutions in homogeneous spaces.

The study of discrete integrable systems is rather new. It began with discretising continuous integrable systems in 1970s. The most well known discretization of the Korteweg-de Vries equation (KdV) is the Toda lattice [45]

\[
\frac{d^2 u_s}{dt^2} = \exp(u_{s-1} - u_s) - \exp(u_s - u_{s+1}).
\]

Here the dependent variable \(u\) is a function of time \(t\) and discrete variable \(s \in \mathbb{Z}\). We can obtain a finite-dimensional version by picking \(N \in \mathbb{N}\) and restricting to \(1 \leq s \leq N\) subject to one of two types of boundary conditions: open-end (\(u_0 = u_N = 0\)) or periodic (\(u_{s+N} = u_s\) for all \(s\) and some period \(N\)). In this paper, we consider the equations defined on infinite lattices, i.e. \(s \in \mathbb{Z}\) for both \(N\)-periodic and non-periodic cases.

Using the Flaschka [8, 9] coordinates

\[
q_s = \frac{du_s}{dt}, \quad p_s = \exp(u_s - u_{s+1}),
\]

we rewrite the Toda lattice (1) in the form

\[
\begin{align*}
\frac{dp_s}{dt} &= p_s(q_s - q_{s+1}), \\
\frac{dq_s}{dt} &= p_{s-1} - p_s.
\end{align*}
\]

Its complete integrability was first established by Flaschka and Manakov [8, 9, 29]. They constructed the Lax representation of system (2) and further solved it by the inverse scattering method.

Another famous integrable discretization of the KdV equation is the Volterra lattice [29, 22]

\[
\frac{dq_s}{dt} = q_s(q_{s+1} - q_{s-1}).
\]

By the Miura transformation \(q_s = p_s p_{s-1}\), it is related to the equation

\[
\frac{dp_s}{dt} = p_s^2(p_{s+1} - p_{s-1}),
\]

which is the modified Volterra lattice, an integrable discretization of the modified KdV equation.

\(^1\)A sequence of moving frames was also used in [24] to minimize the accumulation of errors in an invariant numerical method.
Since the establishment of their integrability, a great deal of work has been contributed to the study of their other integrable properties including Hamiltonian structures, higher symmetry flows and $r$-matrix structures, as well as to the establishment of integrability for other systems. Some historical background and the development of the theory of discrete integrable systems can be found in [44]. Some classification results for such integrable systems including the Toda and Volterra lattice were obtained by the symmetry approach [47].

In this paper, we introduce the concept of discrete moving frames, and we prove theorems analogous to the classical results of the continuous case: generating properties of Maurer-Cartan invariants, a replacement rule, recursion formulas, general formulas for invariant evolutions of polygons, and so on. Once the groundwork is in place, we study the evolution induced on the Maurer-Cartan invariants by invariant evolutions of $N$-gons, the so-called invariantizations. We will show that the invariantization of certain time evolutions of $N$-gons (or so-called twisted $N$-gons in the periodic case) in the centro-affine plane and the projective line $\mathbb{RP}^1$ naturally lead to Hamiltonian pairs. Under the Miura transformations, we can transform the Hamiltonian pairs into the known Hamiltonian pairs for the Toda [1] and modified Volterra [27] lattices respectively. We will also analyze in detail the case of the 2-homogeneous sphere. We will use normalization equations to obtain Maurer-Cartan invariants and we will prove that they are the classical discrete arc-lengths (the length of the arcs joining vertices) and the discrete curvatures ($\pi$ minus the angle between two consecutive sides of the polygon). We will then write the general formula for invariant evolutions of polygons on the sphere and their invariantizations. We finally find an evolution of polygons whose invariantization is a completely integrable evolution of Volterra type.

The arrangement of the paper is as follows: In Section 2 we introduce moving frames and the definitions and calculations we will discretize. In Section 3 we introduce discrete moving frames, discrete Maurer-Cartan invariants and prove our Theorems, including the description of invariant maps and their invariantizations using moving frames. In Section 4 we begin our study of Lie group invariant discrete evolutions of $N$-gons in a homogeneous space, and study the centro-affine and the projective cases, including the completely integrable evolutions of polygons and their associated biHamiltonian pair. We briefly describe invariant maps and their invariantizations and demonstrate that the invariant map in centro-affine plane leads to the integrable discretization of the Toda lattice (2). We'll leave their thorough study, including a geometric interpretation of integrable maps and more examples of maps with a biPoisson invariantization, for a later paper. Section 5 describes the more involved example, that of discrete evolutions on the homogeneous sphere. We conclude with indications of future work.

2. Background and definitions

We assume a smooth Lie group action on a manifold $M$ given by $G \times M \rightarrow M$. In the significant examples we discuss in the later sections, $M$ will be a homogeneous space, that is, $M = G/H$, where $H$ is a closed Lie subgroup, with the standard action.

2.1. Moving Frames. We begin with an action of a Lie Group $G$ on a manifold $M$. 

Definition 2.1. A *group action* of $G$ on $M$ is a map $G \times M \to M$, written as $(g, z) \mapsto g \cdot z$, which satisfies either $g \cdot (h \cdot z) = (gh) \cdot z$, called a *left action*, or $g \cdot (h \cdot z) = (hg) \cdot z$, called a *right action*.

We will also write $g \cdot z$ as $\tilde{z}$ to ease the exposition in places.

Further, we assume the action is *free* and *regular* in some domain $U \subset M$, which means, in effect, that:

1. the intersection of the orbits with $U$ have the dimension of the group $G$ and further foliate $U$;
2. there exists a surface $K \subset U$ that intersects the orbits of $U$ transversally, and the intersection of an orbit of $U$ with $K$ is a single point. This surface $K$ is known as the *cross-section* and has dimension equal to $\dim(M) - \dim(G)$;
3. if we let $O(z)$ denote the orbit through $z$, then the element $h \in G$ that takes $z \in U$ to $k$, where $\{k\} = O(z) \cap K$, is unique.

![Figure 1](image.png)

**Figure 1.** The definition of a right moving frame for a free and regular group action. It can be seen that $\rho(g \cdot z) = \rho(z)g^{-1}$ (for a left action). A left moving frame is obtained by taking the inverse of $\rho(z)$.

Under these conditions, we can make the following definitions.

**Definition 2.2 (Moving frame).** Given a smooth Lie group action $G \times M \to M$, a *moving frame* is an equivariant map $\rho : U \subset M \to G$. We say $U$ is the domain of the frame.

Given a cross-section $K$ to the orbits of a free and regular action, we can define the map $\rho : U \to G$ to be the unique element in $G$ which satisfies

$$\rho(z) \cdot z = k, \quad \{k\} = O(z) \cap K,$$

see Figure 1. We say $\rho$ is the *right moving frame* relative to the cross-section $K$, and $K$ provides the normalization of $\rho$. This process is familiar to many readers: it is well known that if we translate a planar curve so that a point $p$ in the curve is moved to the origin, and we rotate it so that the curve is tangent to the $x$-axis, the second term in the Taylor expansion at $p$ is the Euclidean curvature at $p$. The element of the Euclidean group taking the curve to its normalization is indeed a right moving frame.

By construction, we have for a left action that $\rho(g \cdot z) = \rho(z)g^{-1}$, and for a right action that $\rho(g \cdot z) = g^{-1}\rho(z)$, so that $\rho$ is indeed equivariant. The cross-section...
\( \mathcal{K} \) is not unique, and is usually selected to simplify the calculations for a given application. A left moving frame is the inverse of a right moving frame. Typically, moving frames exist only locally, that is, in some open domain in \( M \). In what follows, we conflate \( \mathcal{U} \) with \( M \) to ease the exposition. In applications however, the choice of domain may be critical.

In practice, the procedure to find a right frame is as follows:

1. define the cross-section \( \mathcal{K} \) to be the locus of the set of equations \( \psi_i(z) = 0 \), for \( i = 1, \ldots, r \), where \( r \) is the dimension of the group \( G \);
2. find the group element in \( G \) which maps \( z \) to \( k \in \mathcal{K} \) by solving the normalization equations,
   \[
   \psi_i(z) = \psi_i(g \cdot z) = 0, \quad i = 1, \ldots, r.
   \]

Hence, the frame \( \rho \) satisfies \( \psi_i(\rho(z) \cdot z) = 0, \ i = 1, \ldots, r \).

Invariants of the group action are easily obtained.

**Definition 2.3** (Normalized invariants). Given a left or right action \( G \times M \rightarrow M \) and a right frame \( \rho \), the normalized invariants are the components of \( I(z) = \rho(z) \cdot z \).

Indeed, for a left action we have
\[
I(g \cdot z) = \rho(g \cdot z) \cdot g \cdot z = \rho(z) g^{-1} g \cdot z = \rho(z) \cdot z = I(z).
\]

For a left frame, the normalized invariants are \( I(z) = \rho(z)^{-1} \cdot z \).

The normalized invariants are important because any invariant can be written in terms of them; this follows from the following:

**Theorem 2.4** (Replacement Rule). If \( F(z) \) is an invariant of the action \( G \times M \rightarrow M \), and \( I(z) \) is the normalized invariant for a moving frame \( \rho \) on \( M \), then \( F(z) = F(I(z)) \).

The theorem is proved by noting that (for a right frame) \( F(z) = F(g \cdot z) = F(\rho(z) \cdot z) = F(I(z)) \) where the first equality holds for all \( g \in G \) as \( F \) is invariant, the second by virtue of setting \( g = \rho(z) \), and the third by the definition of \( I(z) \).

The Replacement Rule shows that the normalized invariants form a set of generators for the algebra of invariants. Further, if we know a sufficient number of invariants, for example, they may be known historically or through physical considerations, then the Replacement Rule allows us to calculate the normalized invariants without knowing the frame.

We can apply this theory to product (also called diagonal) actions, the action with which our paper is concerned. This move has its shortcomings, as we see next.

Given a Lie group action \( G \times M \rightarrow M \), \( (g, z) \mapsto g \cdot z \), the product action is
\[
G \times (M \times M \times \cdots \times M), \quad (g, (z_1, z_2, \ldots, z_N)) \mapsto (g \cdot z_1, g \cdot z_2, \ldots, g \cdot z_N).
\]

In this case the normalized invariants are the components of \( I(z) = (I(z_1), I(z_2), \ldots, I(z_N)) \), and the Replacement Rule for the invariant \( F(z_1, z_2, \ldots, z_N) \) has the form:
\[
F(z_1, z_2, \ldots, z_N) = F(I(z_1), I(z_2), \ldots, I(z_N))
\]

These were called joint invariants in [37].

The following simple scaling and translation group action on \( \mathbb{R} \) will be developed as our main expository example, as the calculations are easily seen.

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\(^2\)Although our theory does not preclude actions on more general product spaces, in the applications we consider, all the spaces comprising the product will be the same.
Example 2.5.
Let \( G = \mathbb{R}^+ \ltimes \mathbb{R} = \{ (\lambda, a) \mid \lambda > 0, a \in \mathbb{R} \} \) act on \( M = \mathbb{R} \) as
\[
z \mapsto \lambda z + a.
\]
The product action is then given by \( z_n \mapsto \lambda z_n + a \) for all \( N \). We write this action as a left linear action as
\[
\left( \begin{array}{c} z_1 \\ z_2 \\ \vdots \\ z_N \end{array} \right) \mapsto \left( \begin{array}{c} \lambda \\ a \end{array} \right) \left( \begin{array}{c} z_1 \\ z_2 \\ \vdots \\ z_N \end{array} \right).
\]
There are two group parameters and so we need two independent normalization equations. We may set
\[
g \cdot z_1 = 0, \quad g \cdot z_2 = 1
\]
which are solvable if \( z_1 \neq z_2 \), and this then defines the domain of this frame. In matrix form the right frame is,
\[
\rho (z_1, z_2, \ldots, z_N) = \left( \begin{array}{cc} \frac{-1}{z_1 - z_2} & \frac{z_1}{z_1 - z_2} \\ 0 & \frac{1}{z_1 - z_2} \end{array} \right).
\]
Equivariance is easily shown. For \( h = h(\mu, b) \):
\[
\rho (h \cdot z_1, h \cdot z_2, \ldots, h \cdot z_N) = \left( \begin{array}{cc} \frac{-1}{\mu(z_1 - z_2)} & \frac{\mu z_1 + b}{\mu(z_1 - z_2)} \\ 0 & \frac{1}{\mu(z_1 - z_2)} \end{array} \right)
\]
\[
= \left( \begin{array}{cc} \frac{-1}{z_1 - z_2} & \frac{z_1}{z_1 - z_2} \\ 0 & \frac{1}{z_1 - z_2} \end{array} \right) \left( \begin{array}{cc} \frac{1}{\mu} & -\frac{b}{\mu} \\ 0 & \frac{1}{\mu} \end{array} \right)
\]
\[
= \rho (z_1, z_2, \ldots, z_N) h(\mu, b)^{-1}
\]
The normalized invariants are then the components of
\[
\left( \begin{array}{c} \frac{-1}{z_1 - z_2} & \frac{z_1}{z_1 - z_2} \\ 0 & \frac{1}{z_1 - z_2} \end{array} \right) \left( \begin{array}{c} I(z_1) \\ I(z_2) \\ \vdots \\ I(z_N) \end{array} \right)
\]
\[
= \left( \begin{array}{c} I(z_1) \\ I(z_2) \\ \vdots \\ I(z_N) \end{array} \right)
\]
\[
= \left( \begin{array}{c} 0 \\ \frac{z_3 - z_1}{z_2 - z_1} \\ \vdots \\ \frac{z_N - z_1}{z_2 - z_1} \end{array} \right),
\]
noting that \( I(z_1) = 0 \) and \( I(z_2) = 1 \) are the normalization equations. Any other invariant can be written in terms of these by the Replacement Rule. For example, it is easily verified that the invariant
\[
\left( \frac{z_5 - z_1}{z_N - z_6} \right)^2 = \left( \frac{I(z_5) - I(z_4)}{I(z_N) - I(z_6)} \right)^2 = \left( \frac{z_5 - z_1}{z_2 - z_1} - \frac{z_4 - z_1}{z_2 - z_1} \right)^2.
\]
The above example shows both the power and the limits of the moving frame.
In applications involving discrete systems, use of the shift operator $\mathcal{T}$ taking $z_n$ to $z_{n+1}$ is central to the calculations and formulae involved. However, it can be seen in the above example that $\mathcal{T}(I(z_n)) \neq I(z_{n+1})$ and that instead the Replacement Rule will lead to complicated expressions for $\mathcal{T}(I(z_n))$, rendering the use of the moving frame at best awkward for applications. This is, in effect, a central computational problem solved by our discrete moving frame.

As motivation the next example will show how an indexed sequence of moving frames arises naturally in a discrete variational problem. Such a sequence is an example of our discrete moving frame.

Example 2.6.

Consider the discrete variational problem

$$L[z] = \sum L_n(z) = \sum \frac{1}{2} I^2_n = \sum \frac{1}{2} \left( \frac{z_{n+2} - z_{n+1}}{z_{n+1} - z_n} \right)^2$$

for which it is desired to find $(z_1, z_2, \ldots, z_N)$ which minimises $L[z]$ subject to certain boundary conditions. We note that $I_{n+k} = (z_{n+k+2} - z_{n+k+1})/(z_{n+k+1} - z_{n+k})$ is invariant under the scaling and translation action, $z_n \mapsto \lambda z_n + a$ (see Example 2.5) and that $\mathcal{T}(I_{n+k}) = I_{n+k+1}$ where $\mathcal{T}$ is the shift operator. The discrete Euler Lagrange equation is

$$0 = \frac{\partial L_n}{\partial z_n} + \mathcal{T}^{-1} \frac{\partial L_n}{\partial z_{n+1}} + \mathcal{T}^{-2} \frac{\partial L_n}{\partial z_{n+2}}$$

which can be written (after shifting to balance the indices) as

$$0 = I^2_{n+1} - I^2_n - I^3_n + I_n I^2_{n-1}.$$

The Lie group invariance of the summand $L_n$ implies there are two conservation laws (in this case, first integrals) arising from the discrete analogue of Noether’s Theorem [15, 21]. These can be written in matrix form as

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = A_n(z) v_n(I) = \begin{pmatrix} -1 \\ z_{n+1} - z_n \end{pmatrix} \begin{pmatrix} 0 \\ -z_{n+1} + b/\mu \end{pmatrix}$$

$$+ \begin{pmatrix} I^2_{n+1} - I^2_n \\ I_n \\ I^2_n \end{pmatrix} \begin{pmatrix} 0 \\ -z_{n+1} - z_n \end{pmatrix} \begin{pmatrix} 0 \\ -z_{n+1} - z_n \end{pmatrix},$$

where this defines the matrix $A_n(z)$ and the vector of invariants $v_n(I)$. It can be seen that the Euler Lagrange equation gives a recurrence relation for $I_n$ and that once this is solved, the conservation laws yield (if $c_1 \neq 0$),

$$z_{n+1} = \frac{c_2 - I^2_n}{c_1}.$$

The matrix $A_n(z)$ is equivariant under the group action, for each $N$, indeed

$$A_n(g \cdot z) = \begin{pmatrix} -\frac{1}{\mu(z_{n+1} - z_n)} & 0 \\ -z_{n+1} + b/\mu & z_{n+1} - z_n \end{pmatrix} \begin{pmatrix} \frac{1}{\mu} & 0 \\ \frac{b}{\mu} & 1 \end{pmatrix} \begin{pmatrix} -\frac{1}{z_{n+1} - z_n} & 0 \\ -z_{n+1} - z_n & 1 \end{pmatrix},$$

and hence each $A_n(z)$ is a left moving frame for the group $G = \mathbb{R}^+ \ltimes \mathbb{R}$, albeit for a different representation for this group. In fact, this is the Adjoint representation.
of $G$, placing this result in line with Theorems on the equivariance of Noether’s conservation laws for smooth systems [11, 30].

![Diagram](image1)

**Figure 2.** The location of the arguments occurring in the normalization equations for a moving frame for the product action on $M \times M \times \cdots \times M$, shown here as disjoint copies of $M$ for clarity, for the Example 2.5.

![Diagram](image2)

**Figure 3.** The sequence of moving frames for the product action on $M \times M \times \cdots \times M$ in Example 2.6, shown here as disjoint copies of $M$ so the location of the arguments occurring in the normalization equations can be easily seen. For this example, each $K_i$ is a shift of the previous. This sequence of moving frames is an example of a discrete moving frame.

In Figure 2 is shown, schematically, the location of the arguments occurring in the normalization equations for the frame of Example 2.5. By contrast, in Figure 3 is shown, schematically, the location of the arguments occurring in the normalization equations for the sequence of frames calculated in Example 2.6. In fact, for that example, each of the $K_i$ will be shifts of each other.

In the next section, we will define a discrete moving frame to be a sequence of moving frames with a nontrivial intersection of domains, and will explore the properties of the structure of the algebra of invariants that arises, in the case where $M$ is a homogeneous manifold and $G$ is represented as a matrix group.
3. Discrete moving frames for twisted $N$-gons in a homogeneous manifold $M = G/H$

3.1. Discrete moving frames and discrete invariants. In this section we will state the definition of discrete group-based moving frame along $N$-gons and show that a parallel normalization process will produce right discrete moving frames. A discrete moving frame gives many sets of generators for the algebra of invariants under the action of the group. We will show recursion relations between these, and find a small, useful set of generators which we will denote as discrete Maurer-Cartan invariants.

In both our theory and our examples, $M$ will be a homogeneous space, that is, $M = G/H$, where $H$ is a closed Lie subgroup. For our examples, $G$ is semi-simple, although this is not needed in general. The following definition appeared in [39].

**Definition 3.1 (Twisted $N$-gon).** A twisted $N$-gon in a manifold $M$ is a map $\phi : \mathbb{Z} \to M$ such that for some fixed $g \in G$ we have $\phi(k + N) = g \cdot \phi(k)$ for all $k \in \mathbb{Z}$. (Recall $\cdot$ represents the action of $G$ on $M$.) The element $g \in G$ is called the monodromy of the gon.

We will denote by $\mathcal{P}_N$ the space of twisted $N$-gons in $M$ and we will denote a twisted $N$-gon by its image $(x_r)$ where $x_r = \phi(r)$. That is, $\mathcal{P}_N$ can be identified with the Cartesian product of $N$ copies of the manifold $M$, which we will denote by $M^{(N)}$. The group $G$ acts naturally on $\mathcal{P}_N$ using the induced action of $G$ on the manifold $M$, that is, using the diagonal or product action of $G$ on $M^{(N)}$. Twisted $N$-gons will give rise to $N$-periodic invariants, but for much of what follows we can also consider infinite gons. We will distinguish these two cases as needed in the applications.

Our next definition represents the discrete analog of the group-based moving frame we described in Section 2.

**Definition 3.2 (Discrete moving frame).** Let $G^{(N)}$ denote the Cartesian product of $N$ copies of the group $G$. Allow $G$ to act on the left on $G^{(N)}$ using the diagonal action $g \cdot (g_r) = (gg_r)$. We also consider what we call the “right inverse action” $g \cdot (g_r) = (g_r g^{-1})$. (We note these are both left actions according to Definition (2.1).) We say a map

$$\rho : \mathcal{P}_N \to G^{(N)}$$

is a left (resp. right) discrete moving frame if $\rho$ is equivariant with respect to the action of $G$ on $\mathcal{P}_N$ and the left (resp. right inverse) action of $G$ on $G^{(N)}$. Since $\rho((x_r)) \in G^{(N)}$, we will denote by $\rho_r$ its $r$th component, that is $\rho = (\rho_r)$, where $\rho_s((x_r)) \in G$ for all $s$. Equivariance means,

$$\rho_s(g \cdot (x_r)) = \rho_s((g \cdot x_r)) = g \rho_s((x_r)) \quad (\text{resp. } \rho_s((x_r)) g^{-1})$$

for every $s$. Clearly, if $\rho = (\rho_r)$ is a left moving frame, then $\hat{\rho} = (\rho_r^{-1})$ is a right moving frame.

As in the continuous case, one can obtain right discrete moving frames using a normalization process.

**Standing assumption** In this paper, we give proofs of our Theorems for the simplest kind of normalization equations, those where (some) components of the $x_r$ have been normalized to constants. Hence we assume this at the outset and leave the generalization to be developed elsewhere.
Theorem 3.3. Assume ρ = (ρr((x,r)))) is uniquely determined by equations of the form
\[ \rho_s((x,r)) \cdot x_k = c_k^s, \] (4)
for some s, k, where c_k^s ∈ R are called normalization constants. Note that we must have enough equations to determine each ρ_s uniquely, and we assume consistency of the normalization equations with the monodromy condition on the N-gon. Then ρ = (ρr((x,r))) is a right moving frame along the N-gon (x,r).

Proof. The proof is entirely analogous to that of a single frame. We will denote by ρ_s((x,r)) the unique element of the group determined by equations (4). That means ρ_s(g · (x,r)) is the unique element of the group determined by equations
\[ \rho_s(g(x,r)) \cdot (g \cdot x_k) = c_k^s. \] (5)
On the other hand ρ_s((x,r))g^{-1} also satisfies those equations, and so ρ_s(g · (x,r)) = ρ_s((x,r))g^{-1}. Therefore ρ is a right discrete moving frame.

Definition 3.4. Let \( F : \mathcal{P}_N \rightarrow \mathbb{R} \) be a function defined on N-gons. We say that F is a discrete invariant if
\[ F((g \cdot x,r)) = F((x,r)) \] (6)
for any g ∈ G and any (x,r) ∈ \( \mathcal{P}_n \).

Notice that, whether normalized to constants or not, the quantities \( \rho_s((x,r)) \cdot x_k = I_k^s \) are always invariants as we can readily see from \( g \cdot I_k^s = \rho_s(g(x,r)) \cdot (g \cdot x_k) = \rho_s((x,r))g^{-1}g \cdot x_k = \rho_s((x,r)) \cdot x_k = I_k^s \).

Proposition 3.5 (s-Replacement rule). If ρ is a right moving frame, and F((x,r)) is any invariant, then
\[ F((x,r)) = F((\rho_s \cdot x,r)) \] (7)
Further, the \( \rho_s \cdot x_k = I_k^s \) with s fixed, \( k = 1, \ldots, N \) generate all other discrete invariants. We call the \( I_k^s \) with s fixed, the s-basic invariants.

Proof. This is the same as Theorem 2.4 for each s. From (6) we need only choose \( g = \rho_s \) in (6). In particular, Equation (7) means F can be written as a function of the s-basic invariants: one merely needs to substitute \( x_r \) by \( \rho_s \cdot x_r \) for s fixed.

3.2. Discrete Maurer-Cartan invariants. A discrete moving frame on \( \mathcal{P}^N \) provides N different sets of generators, since each \( \rho_s \) generates a complete set. With this abundance of choices one is hard pressed to choose certain distinguished invariants, and a small group that will generate all others. Although a correct choice will depend on what you would like to use the invariants for, a good choice in our case are what we will denote the discrete Maurer-Cartan invariants. They are produced by the discrete equivalent of the Serret-Frenet equations.

Definition 3.6. Let \( \rho_r \) be a left (resp. right) discrete moving frame evaluated along a twisted N-gon, the element of the group
\[ K_r = \rho_r^{-1} \rho_{r+1} \quad (\text{resp. } \rho_{r+1} \rho_r^{-1}) \]
is called the r-Maurer Cartan matrix for \( \rho \). We will call the equation \( \rho_{r+1} = \rho_r K_r \) the discrete r-Serret-Frenet equation.
One can directly check that if $K_r$ is a left Maurer-Cartan matrix for the left frame $(\rho_r)$, then $K_r^{-1}$ is a right one for the right frame $\tilde{\rho} = (\rho_r^{-1})$, and vice versa.

The equivariance of $\rho$ immediately yields that the $K_r$ are invariant under the action of $G$. We show next how the components of the Maurer Cartan matrices generate the algebra of invariants by exhibiting recursion relations between them and the normalized invariants, for our expository example.

**Example 2.5 cont.** Recall the group is $G = \mathbb{R}^+ \ltimes \mathbb{R}$ acting on $\mathbb{R}$ as $\tilde{z} = (\lambda, a) \cdot z = \lambda z + a$, which is represented so that the action is left (multiplication) as

\[
\begin{pmatrix}
\lambda & a \\
0 & 1 \\
\end{pmatrix}
\begin{pmatrix}
z \\
1 \\
\end{pmatrix}
= \begin{pmatrix}
\lambda z + a \\
1 \\
\end{pmatrix}.
\]

If we take the normalization equations of $\rho_r$ to be $\tilde{z}_r = 1$ and $\tilde{z}_{r+1} = 0$, then the right discrete frame is

\[
\rho_r = \begin{pmatrix}
-z_r + 1 & z_r + 1 \\
0 & 1 \\
\end{pmatrix}.
\]

By definition, $\rho_r \cdot z_s = I_s = -(z_s - z_r + 1)/(z_r + 1 - z_r)$ and then

\[K_r = \rho_r \rho_{r+1}^{-1} = \begin{pmatrix}
-I_{r+2} & I_{r+2} \\
0 & 1 \\
\end{pmatrix}
\]

as can be verified directly. We do not need, however, to have solved for the frame to obtain this result; it can be obtained directly by noting that we have both

\[
(\rho_{r+1})(z_{r+1} \quad z_{r+2}) = \begin{pmatrix}
1 & 0 \\
1 & 1 \\
\end{pmatrix}, \quad \rho_r(z_{r+1} \quad z_{r+2}) = \begin{pmatrix}
0 & I_{r+2} \\
1 & 1 \\
\end{pmatrix}
\]

or

\[
\rho_r \rho_{r+1}^{-1} = \begin{pmatrix}
0 & I_{r+2} \\
1 & 1 \\
\end{pmatrix} \begin{pmatrix}
1 & 0 \\
1 & 1 \\
\end{pmatrix}^{-1},
\]

verifying the result in (8). Note that for the second calculation, we are using the invariants $I_s$ symbolically; if we have not solved for the frame, then we may not know what these are.

Calculations similar to those of Equation (9) give recursion relations between the $I_s$. For example, we have both

\[
\rho_r(z_{r+k} \quad z_{r+k+1}) = \begin{pmatrix}
I_{r+k} & I_{r+k+1} \\
1 & 1 \\
\end{pmatrix}, \quad \rho_{r+k}(z_{r+k} \quad z_{r+k+1}) = \begin{pmatrix}
1 & 0 \\
1 & 1 \\
\end{pmatrix}
\]

so that

\[
K_r K_{r+1} \cdots K_{r+k-1} = \rho_r \rho_{r+k}^{-1} = \begin{pmatrix}
I_{r+k} & I_{r+k+1} \\
0 & 1 \\
\end{pmatrix}.
\]

However, we also have by direct calculation using our $K_i$, for example,

\[
K_r K_{r+1} = \begin{pmatrix}
I_{r+2} & I_{r+3} \\
0 & 1 \\
\end{pmatrix}
\]

and thus for $k = 2$ we have $I_{r+3} = I_{r+2}(1 - I_{r+3}^{-1})$. In this way, it is possible to obtain all the $I_s$ from the components of the $K_k$’s, that is, the $I_{r+k+2}^k$. This smaller
set of generating invariants we will denote as the discrete Maurer-Cartan invariants.

Example 3.7.

Our next example is the centro-affine action of SL(2, R) on \( \mathbb{R}^2 \); that is SL(2, R) acts linearly on \( \mathbb{R}^2 \). We may cast the centro-affine plane in the form \( M = G/H \) by identifying \( \mathbb{R}^2 \) with \( SL(2, \mathbb{R})/H \), where \( H \) is the isotropy subgroup of \( e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \).

There is a reason why we are choosing \( e_2 \) rather than \( e_1 \); the reader will readily see the connection to Example 3.8, the projective line. In order to find a moving frame we will use the following normalization equations

\[
\rho_r \cdot x_r = c_r^r = e_2, \quad \rho_r \cdot x_{r+1} = c_{r+1}^r = -|x_r, x_{r+1}| e_1
\]

where \(|x_r, x_{r+1}|\) is the determinant of the two 2-vectors; specifically, the second component of \( \rho_r \cdot x_{r+1} \) being zero is the normalization equation, while the first is determined by the condition that \( \rho_r \in SL(2, \mathbb{R}) \). This results in the left moving frame

\[
\rho_r^{-1} = \left( \begin{array}{cc}
-\frac{1}{|x_r, x_{r+1}|} x_{r+1} \\
x_r
\end{array} \right).
\]

From here, a complete set of generating invariants will be the components of

\[
\rho_r \cdot x_s = \rho_r x_s = \left( \begin{array}{c}
\frac{|x_s, x_r|}{|x_r, x_{r+1}|} \\
\frac{|x_s, x_{r+1}|}{|x_r, x_{r+1}|}
\end{array} \right)
\]

with \( r \) fixed and \( s = 1, \ldots, N \); that is, \( |x_r, x_s|, |x_{r+1}, x_s| \) with \( r \) fixed and \( s = 1, \ldots, N \). To see how the \( r \)-basic invariants will generate the \( k \)-basic invariants it suffices to recognise the relation

\[
|x_s, x_k| = |\rho_r x_s, \rho_r x_k| = \left| \begin{array}{c}
|x_r, x_s| \\
|x_r, x_{r+1}| \\
x_k, x_{r+1}
\end{array} \right| |x_r, x_k| = \frac{1}{|x_r, x_{r+1}|} \left( |x_r, x_s| |x_k, x_{r+1}| - |x_r, x_k| |x_s, x_{r+1}| \right)
\]

obtained by the Replacement Rule, Proposition 3.5. The left Maurer-Cartan matrix is given by

\[
(10) \quad \rho_r \rho_r^{-1} = \begin{pmatrix} k^2_r & -k^1_r \\ \frac{k^1_r}{k^2_r} & 0 \end{pmatrix}
\]

where \( k^2_r = \frac{|x_r, x_{r+2}|}{|x_{r+1}, x_{r+2}|} \) and \( k^1_r = |x_r, x_{r+1}| \). By calculations similar to those of the previous example, we have that these invariants, with \( r = 1, \ldots, N \) generate all other invariants. This follows from observing

\[
\rho_r \cdot x_s = \rho_r \rho_r^{-1} \rho_{r+1} \rho_{r+1}^{-1} \cdots \rho_{s-1} \rho_{s-1}^{-1} (\rho_s \cdot x_s) = K_r K_{r+1} \cdots K_{s-1} e_2
\]

and similarly if \( s < r \). In fact, using \( k^i_{r+1} \) we could simplify these generators to the simpler set \( |x_r, x_{r+1}|, |x_r, x_{r+2}|, r = 1, \ldots, N \), as expected.

Our final expository example before proving our result concerning the discrete Maurer Cartan invariants is the projective action of SL(2, R) on \( N \)-gons in \( \mathbb{R}P^1 \).

Example 3.8.
Consider local coordinates in \( \mathbb{RP}^1 \) such that a lift from \( \mathbb{RP}^1 \) to \( \mathbb{R}^2 \) is given by \( x \rightarrow \left( \frac{x}{1} \right) \). In that case \( \mathbb{RP}^1 \) can be identified with \( \text{SL}(2, \mathbb{R})/H \), where \( H \) is the isotropy subgroup of \( x = 0 \). The action is given by the fractional transformations

\[
\rho_r \cdot x_s = \begin{pmatrix} a_r & b_r \\ c_r & d_r \end{pmatrix} \cdot x_s = \frac{a_r x_s + b_r}{c_r x_s + d_r}, \quad a_r d_r - b_r c_r = 1.
\]

To find a discrete right moving frame along a gon \((x_r)\) we will use normalization equations

\[
\rho_r(x_r) = 0, \quad \rho_r(x_{r+1}) = 1, \quad \rho_r(x_{r+2}) = -1
\]

so that the normalization constants are \( c_r^r = 0, c_r^{r+1} = 1 \) and \( c_r^{r+2} = -1 \) for all \( r \), specifically, we solve

\[
\begin{align*}
(a_r x_r + b_r) \quad & = 0, \\
(c_r x_r + d_r) & = 1,
\end{align*}
\]

for \( a_r, b_r, c_r \) noting that \( a_r d_r - b_r c_r = 1 \). The domain of the frame we construct requires that the gon does not close, that is \( x_r \neq x_s \) for \( r \neq s \), and we obtain

\[
\rho_r = a_r \begin{pmatrix} 1 & -x_r \\ x_{r+1} - x_r & x_{r+2} - 2x_{r+1} + x_r \\ x_{r+1} - x_r & x_{r+2} - 2x_{r+1} + x_r \end{pmatrix}, \]

where

\[
a_r^2 = \frac{1}{2} \frac{x_{r+2} - x_{r+1}}{(x_{r+1} - x_r)(x_{r+2} - x_r)}
\]

Thus we have a moving frame assuming the expression above is nonnegative. If the expression is negative we will need to choose different normalization constant, for example, we can simply change \( c_r^{r+2} = \frac{1}{2} \) to cover the negative case.

As in the previous examples, we can produce formulas relating \( \rho_r \cdot x_s, \ s = 0, \ldots, n-1 \) to \( \rho_k \cdot x_l, \ l = 1, \ldots, N \) for two given values \( r \) and \( k \) by the Replacement rule, that is, by substituting \( x_r \) by \( \rho_k \cdot x_r \) in the expression of \( \rho_r \cdot x_s \).

Continuing with this projective case, assume the right Maurer-Cartan matrix \( K_r = \rho_{r+1} \rho_{r}^{-1} \) associated to this right moving frame is given by

\[
K_r = \begin{pmatrix} k_1^r & k_2^r \\ k_3^r & k_4^r \end{pmatrix}.
\]

From \( \rho_r \cdot x_{r+1} = 1 \) and \( \rho_{r+1} \cdot x_{r+1} = 0 \) we obtain \( K_r \cdot 1 = 0 \) and similarly \( K_r \cdot (-1) = 1 \) and \( K_r \cdot I^{r+3} = -1 \). (Recall \( \rho_r \cdot x_{r+3} = I^{r+3} \).) We thus solve

\[
\begin{align*}
\frac{k_1^r + k_2^r}{k_3^r + k_4^r} & = 0, \\
\frac{-k_1^r + k_2^r}{-k_3^r + k_4^r} & = 1, \\
\frac{k_1^r I_{r+3}^r + k_2^r}{k_3^r I_{r+3}^r + k_4^r} & = -1
\end{align*}
\]

together with \( \det(K_r) = 1 \) to obtain the Maurer-Cartan matrix

\[
K_r = \begin{pmatrix} k_r & -k_r \\ 2k_r^2 + 1 & -2k_r^2 - 1 \end{pmatrix}, \quad \text{where} \quad I_{r+3}^r = \frac{4k_r^2 - 1}{4k_r^2 + 1}.
\]

In this particular case one can find a simpler moving frame without resorting to normalization equations. Indeed, we can lift \( x_r \) to \( V_r \in \mathbb{R}^2 \) so that \( \det(V_{r+1}, V_r) = 1 \).
for all \( r \). It suffices to define \( V_r = t_r \begin{pmatrix} x_r \\ 1 \end{pmatrix} \) and solve
\[
t_r t_{r+1} \det \begin{pmatrix} x_r & x_{r+1} \\ 1 & 1 \end{pmatrix} = 1.
\]

If \( N \) is not even, this equation can be uniquely solved for \( t_r, r = 1, \ldots, N \), using the twisted condition \( t_{N+s} = t_s \) for all \( s \). The element
\[
\rho_r = (V_{r+1}, V_r)
\]
is clearly a moving frame since the lift of the projective action to \( \mathbb{R}^2 \) is the linear action. It also satisfies \( \rho_r \cdot o = x_r \), where \( o = (0,1)^T \) is the equivalence class of \( H \) in \( \text{SL}(2,\mathbb{R})/H \), where recall \( H \) is the isotropy subgroup of \( x = 0 \). Given that \( \mathbb{R}^2 \) is generated by \( V_r, V_{r+1} \) for any choice of \( r \), we have that
\[
V_{r+2} = k_r V_{r+1} - V_r
\]
for all \( r \) (the coefficient of \( V_r \) reflects the fact that \( \det(V_{r+1}, V_r) = 1 \) for all \( r \)). From here
\[
\rho_{r+1} = \rho_r \begin{pmatrix} k_r & 1 \\ -1 & 0 \end{pmatrix}
\]
and so the Maurer-Cartan matrix is given by
\[
K_r = \begin{pmatrix} k_r & 1 \\ -1 & 0 \end{pmatrix}.
\]

This invariant appeared in [39].

We now come to the proof that the Maurer-Cartan invariants are generators under assumptions that will hold in the sequel, and leave generalizations to be developed elsewhere. The assumptions are that \( M = G/H \) where \( H \) is a closed subgroup of \( G \) and \( G \) acts on \( M \) in the standard way, that is, by left product on representatives of the class. We assume further that the normalization equations consist of setting certain co-ordinates to constants, and that \( \rho_r \cdot o = x_r \) (for \( \rho_r \) a left frame) where \( o \) is the equivalence class of \( H \) in \( G/H \). We will first demonstrate so-called recursion equations linking the Maurer-Cartan invariants to the full set of basic invariants. Additionally the recursion relations will allow us to solve for the matrices \( K_r \) directly from the normalization equations without any need for an explicit moving frame. The essence of the proof has already been demonstrated in the examples.

From now on we will assume that \( G \subset \text{GL}(m,\mathbb{R}) \) and so \( K_r \) are represented by matrices; this is not necessary, but it is convenient.

**Theorem 3.9.** Let \( K = (K_r) \) be the left (resp. right) Maurer-Cartan matrix associated to a moving frame \( (\rho_r) \) determined by normalization constants \( c^r_s \). Then \( K \) satisfies the equation
\[
K_r \cdot c^r_{s+1} = c^r_s \quad (\text{resp. } K_r \cdot c^r_s = c^r_{s+1})
\]
for all \( s, r \).

**Proof.** The proof is direct. If \( \hat{\rho}_r \) is the right moving frame determined by normalization constants \( c^r_s \), then \( \rho_r = \hat{\rho}_{r-1} \) will define a left moving frame and will satisfy
\[
\rho_r \cdot c^r_s = x_s.
\]
Since $K_r = \rho_r^{-1}\rho_{r+1}$, $K_r$ will satisfy
\[ K_r \cdot e_s^{r+1} = \rho_r^{-1} \cdot x_s = e_s^r \]
as stated. Since the right Maurer-Cartan matrix for the right frame $\hat{\rho}_r$ is the inverse of the left one for $\hat{\rho}_r^{-1}$, it will satisfy $K_r \cdot e_s^r = e_s^{r+1}$.

Notice that, in general, if we denote $\rho_r \cdot x_s = I_s^r$, whether the invariant $I_s^r$ is normalized or not, the recursion relation still holds true. That is, the recursion formula looks like
\[ K_r \cdot I_s^{r+1} = I_s^r \]
with some $I_s^r$ normalized and some not. If the normalization equations guarantee that $\rho$ is uniquely determined, a rank argument will equally guarantee that $K = (K_r)$ is uniquely determined by equations as above. Using the recursion formula we can prove the generating property.

**Corollary 3.10.** Assume $c_r^s = o$ for all $r$, where $o$ is the class of $H$ in $G/H = M$. Let $I$ be a discrete invariant for the $N$-gon $(x_r)$. Then $I$ is a function of the entries of $K_r$, $r = 1, \ldots, N$.

**Proof.** First of all, we know that $I$ will be a function of the basic invariants $I_s^r$ by the Replacement Rule, so it suffices to show that $I_s^r$ are all generated by the entries of $K_r$. And this directly follows from the recursion formula. Indeed, we can use the recursion formulas for $K_r, K_{r+1}, \ldots, K_{n-1}$ to find $I_1^{r+1}, \ldots, I_r^{r+1}$ from the values $I_r^s = c_r^s = o$ recursively. If we invert the recursion equation to read
\[ I_s^{r+1} = K_r^{-1} \cdot I_s^r \]
then we can use the equations for $K_r, K_{r+1}, \ldots, K_{n-1}$ to equally generate $I_r^{r+1}$, $I_{r+1}^r, \ldots, I_n^r$ from the value $I_s^r = o$. Therefore, $K_r$ will generate $I_s^r$ for any $s, r = 1, \ldots, N$.

We will finally show that given $K_r$, and assuming $\rho_r \cdot o = x_r$, for the moving frames generating $K_r$, the $N$-gon is completely determined up to the action of the group.

**Proposition 3.11.** Let $(x_r)$ and $(\hat{x}_r)$ be two twisted $N$-gons with left moving frames $\rho$ and $\hat{\rho}$ such that $\rho_r \cdot o = x_r$, $\hat{\rho}_r \cdot o = \hat{x}_r$ and $\rho_r^{-1}\rho_{r+1} = \hat{\rho}_r^{-1}\hat{\rho}_{r+1} = K_r$. Then, there exists $g \in G$ such that $x_r = g\hat{x}_r$ for all $r$.

**Proof.** Notice that from $\rho_{r+1} = \rho_r K_r$ and $\hat{\rho}_{r+1} = \hat{\rho}_r K_r$ it suffices to show that there exists $h \in H$ such that $\rho_0 = h\rho_0$. Indeed, if this is true we will have $\rho_r = h\rho_r$ for all $r$, and from here $\rho_r \cdot o = x_r = h\hat{\rho}_r \cdot o = h \cdot \hat{x}_r$.

On the other hand, clearly exists $g \in G$ such that $\rho_0 = g\rho_0^{-1}$, and $g \in H$ since both $\rho_0$ and $\hat{\rho}_0$ leave $o$ invariant. The proposition follows.

4. **Lie symmetric evolutions, maps and their invariantizations**

An evolution equation is said to have a Lie group symmetry if the Lie group action takes solutions to solutions. A recurrence map is said to have a Lie group symmetry if it is equivariant with respect to the action of the group. Such equations and maps are usually called invariant, and we will do so here. However, it is
important to note that this does not mean that the equations are comprised of invariants of the action.

In this section we will show how to write any invariant time evolution of twisted \(N\)-gons, as well as any invariant map from the space of twisted \(N\)-gons to itself, in terms of the invariants and the moving frame, in a straightforward and explicit fashion. We continue to assume that the assumptions of the theorems of the last section hold, so that \(M = G/H\), that \(\rho_r \cdot o = x_r\) (for a left frame).

Assume we have a left moving frame \(\rho\) in the neighbourhood of a regular twisted \(N\)-gon, and let

\[
(x_s)_t = f_s((x_r))
\]

be an invariant time evolution of the twisted \(N\)-gons under the action of the group, that is, if \((x_r)\) is a solution, so is \((g \cdot x_r)\) for any \(g \in G\). Denote by \(\Phi_g : M \rightarrow M\) the map defined by the action of \(g \in G\) on \(M\), that is \(\Phi_g(x) = g \cdot x\).

**Theorem 4.1.** Any evolution of the form (14) can be written as

\[
(x_s)_t = d\Phi_{\rho_s}(o)(v_s)
\]

where \(o = [H]\), and \(v_s((x_r)) \in T_{x_r}M\) is an invariant vector, that is, \(v_s((g \cdot x_r)) = v_s((x_r))\) for any \(g \in G\) and for all \(s\).

**Proof.** Assume we have an invariant evolution of the form (14). To prove this statement we merely need to prove that

\[
d\Phi_{\rho_s((x_r))}(o)^{-1} f_s((x_r))
\]

is invariant under the action of \(G\), and call it \(v_s\). Indeed, since (14) is an invariant evolution,

\[
(g \cdot x_s)_t = d\Phi_g(x_s) \cdot (x_s)_t = d\Phi_g(x_s) \cdot f_s((x_r)) = f_s((g \cdot x_r)).
\]

Also, since \(\rho\) is a left moving frame with \(\rho_s \cdot o = x_s\)

\[
d\Phi_{\rho_s((x_r))}(o) = d\Phi_{g\rho_s((x_r))}(o) = d\Phi_g(x_s)d\Phi_{\rho_s((x_r))}(o).
\]

From here

\[
d\Phi_{\rho_s((x_r))}(o)d\Phi_{\rho_s((x_r))}^{-1}(o)f_s((x_r)) = f_s((g \cdot x_r)),
\]

which implies that \(d\Phi_{\rho_s((x_r))}(o)^{-1} f_s((x_r))\) is invariant. \(\square\)

Consider now an invariant map of the form

\[
T(x_s) = g_s((x_r)),
\]

so by definition \(T(g \cdot x_s) = gT(x_s)\) for any \(g \in G\). We have the following theorem.

**Theorem 4.2.** If \(T\) is an invariant map of the form (16), then

\[
T(x_s) = \rho_s((x_r)) \cdot z_s((x_r))
\]

where \(z_s((x_r)) \in M\) is an invariant element of \(M\), that is, \(z_s((g \cdot x_r)) = z_s((x_r))\) for any \(g \in G\).

**Proof.** As before, it suffices to show that \(\rho_s^{-1} g_s((x_r))\) is invariant and to call it \(z_s\). And as before this is straightforward since

\[
\rho_s((g \cdot x_r)) = g\rho_s((x_r))
\]

and

\[
T(g \cdot x_s) = g_s((g \cdot x_r)) = gT(x_s) = g g_s((x_r)),
\]

\[
T(x_s) = \rho_s((x_r)) \cdot z_s((x_r))
\]
so that
\[
\rho_s^{-1}((g \cdot x_r))g_s((g \cdot x_r)) = \rho_s^{-1}((x_r))g_s((x_r)).
\]

If either a map or a time evolution is invariant, then there is a corresponding induced map or evolution on the invariants themselves. The reduction process is at times very involved and time consuming. Here we will describe a simple and straightforward way to find explicitly this so-called invariantization of the evolution. Further, in the examples we detail how, for a proper choice of evolutions, the resulting invariantizations are integrable, in the sense that they can be written in two different ways as a Hamiltonian system, using a Hamiltonian pair. We also illustrate, in Example 4.10, that some invariantized map also results in a discrete evolution.

Assumption. From now on we will assume to have chosen \( \varsigma: M = G/H \to G \), a section of the quotient \( G/H \) such that \( \varsigma(o) = e \in G \), where \( e \) is the identity.

**Theorem 4.3.** Assume we have an invariant evolution of the form (15) and let \( \varsigma \) be a section such that \( \varsigma(o) = e \in G \). Assume \( \rho_r \cdot o = x_r \) and \( \rho_r = \varsigma(x_r)\rho_r^H \), where \( \rho_r^H \in H \). Then
\[
(K_s) \cdot (\rho_s^{-1}(\rho_{s+1})) = \rho_s^{-1}(\rho_{s+1}) - \rho_s^{-1}(\rho_s)\rho_{s+1}^{-1} = K_sN_{s+1} - N_sK_s.
\]

Proof. The first part of the proof is a straightforward computation
\[
(K_s) = (\rho_s^{-1}(\rho_{s+1})) = \rho_s^{-1}(\rho_{s+1}) - \rho_s^{-1}(\rho_s)\rho_{s+1}^{-1} = K_sN_{s+1} - N_sK_s.
\]

Before proving the second part we notice that is \( \varsigma \) is a section, then
\[
g \varsigma(x) = \varsigma(g \cdot x)h(x, g)
\]
for some unique \( h \in H \). In fact, one can take this relation as defining uniquely the action of the group \( G \) on a homogeneous space \( G/H \) in terms of the section. We will use this relation shortly.

Since \( \rho_s = \varsigma(x_s)h((x_r)) \), a calculation shows
\[
N_s = \rho_s^{-1}(\rho_s) = \left(L_{\rho_r^{-1}}\right)_* (R_{\rho_r^H})_* d\varsigma(x_s)(x_s) + \left(L_{\rho_r^{-1}}\right)_* d\rho_r^H (\varsigma(x_s))(x_s)\]
\[
\]
where \( L \) and \( R \) signify left and right multiplication and where the asterisk denotes differentiation as customary. Very clearly the second term belongs to \( h \) and so we will focus on the first term. If we differentiate (20) evaluated on \( x_r \), we have
\[
(L_g)_* d\varsigma(x_r)(x_r) = (R_{h(x_r, g)})_* d\varsigma(F_g(x_r))d\Phi_g(x_r)(x_r) + (L_{\varsigma(g \cdot x)})_* dh(x_r, g)(x_r)\]
for any \( g \). If we now substitute \( g = \rho_r^{-1} \) and we use (15), \( \varsigma(o) = e \) and \( d\Phi_{\rho_r^{-1}}(x_r) = (d\Phi_{\rho_r^{-1}}(x_r))^{-1} \) we have
\[
(L_{\rho_r^{-1}})_* d\varsigma(x_r)(x_r) = (R_{h(x_r, \rho_r^{-1})})_* d\varsigma(o)v_r + dh(x_r, \rho_r^{-1})(x_r)\]
The last term is again an element of $H$ and we can ignore it. Finally, from (20) we see that

$$(\rho_r^H)^{-1} = \rho_r^{-1} \zeta(x_r) = \zeta(\rho_r^{-1} \cdot x_r) h(x_r, \rho_r^{-1}) = \zeta(o) h(x_r, \rho_r^{-1}) = h(x_r, \rho_r^{-1}).$$

Therefore

$$\left( R_{h(x_r, \rho_r^{-1})} \right)_a d\zeta(o) \mathbf{v}_r = \left( R_{(\rho_r^H)^{-1}} \right)_a d\zeta(o) \mathbf{v}_r.$$

Going back to the splitting of $N_r$ in (21) we see that

$$N_r^m = \left( R_{\rho_r^H} \right)_a \left( R_{(\rho_r^H)^{-1}} \right)_a d\zeta(o) \mathbf{v}_r = d\zeta(o) \mathbf{v}_r$$

as stated in the theorem.

It is often the case that conditions (19) allow us to solve explicitly for the $N_r$ directly from equation (18), as we will see next in our examples.

**Example 4.4.**

In the centro-affine case (see Example 3.7) the action is linear and so $d\Phi_{\rho_s}(o) \mathbf{v}_s = \rho_s \mathbf{v}_s$. The general invariant evolution is given by

$$(x_s)_t = -\frac{v_1^s}{x_s, x_{s+1}} x_{s+1} + v_2^s x_s = \rho_s^{-1} \left( \begin{array}{c} v_1^s \\ v_2^s \end{array} \right)$$

where $v_1^s, v_2^s$ are arbitrary functions of the invariants previously obtained in (10) (recall that $\rho_s$ was in this case a right moving frame and we need a left one here).

If we want to find the evolution induced on $k_1^s$ and $k_2^s$ as in (10), then we will recall that the space $H$ is the isotropy subgroup of $e_2$, which is the subgroup of strictly lower triangular matrices. A section for the quotient is given by

$$(22) \quad \zeta \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} b^{-1} & a \\ 0 & b \end{bmatrix}$$

if $b \neq 0$ (we work in a neighborhood of $o = e_2$). Clearly $\rho_s = \zeta(x_r) \rho_r^H$ since $\zeta(x_r)^{-1} \rho_s \in H$. A complement $m$ to $h$ is given by the upper triangular matrices. In that case

$$d\zeta(o) \mathbf{v}_s = \left( \begin{array}{c} -v_2^s \\ 0 \\ v_1^s \\ v_2^s \end{array} \right)$$

and so

$$N_s = \begin{bmatrix} -v_2^s & v_1^s \\ 0 & v_2^s \end{bmatrix}$$

where $\alpha_s$ is still to be determined. From equation (18) we have

$$\left( \begin{array}{cc} (k_2^s)_t & -k_1^s \\ (k_1^s)^{-1}_t & 0 \end{array} \right) = \left( \begin{array}{cc} k_2^s & -k_1^s \\ 1 & 0 \end{array} \right) \left( \begin{array}{cc} v_1^s & v_{s+1}^s \\ v_{s+1}^s & v_{s+2}^s \end{array} \right) - \left( \begin{array}{cc} -v_2^s & v_1^s \\ 0 & v_2^s \end{array} \right) \left( \begin{array}{cc} k_2^s & -k_1^s \\ 1 & 0 \end{array} \right)$$

$$= \left( \begin{array}{cc} -k_2^s v_{s+1}^s + k_1^s \alpha_{s+1} + v_2^s k_2^s - \frac{v_1^s}{k_2^s} & k_2^s v_{s+1}^s - k_1^s v_{s+2}^s - v_2^s k_1^s \\ -v_{s+1}^s + \alpha_s k_2^s - \frac{v_1^s}{k_2^s} & \frac{v_{s+1}^s}{k_2^s} + \alpha_s k_1^s \end{array} \right).$$

The entry (2, 2) of this system is given by $0 = k_1^s \alpha_s + (k_1^s)^{-1} v_{s+1}^s$, which allows us to solve for the missing entry

$$\alpha_s = -\frac{v_{s+1}^s}{(k_1^s)^2}.$$
The other entries give us the evolution of the invariants. These are

\[
(k^2_{s+1}) = \begin{pmatrix}
-2k^2_s v^2_{s+1} + v^1_{s+2} \\
-2k^2_s v^2_{s+1} + v^1_{s+2}
\end{pmatrix} := A \begin{pmatrix}
v^2_s \\
v^1_s
\end{pmatrix},
\]

where the matrix difference operator is

\[
A = \begin{pmatrix}
k^1_s(T + 1) & -k^2_s T \\
-k^2_s(T - 1) & -\frac{1}{k^2_s} + \frac{k^1_s}{(k^2_s + 1)^2} T^2
\end{pmatrix}
\]

and where \( T \) is the shift operator \( T a_s = a_{s+1} \).

Let us define a diagonal matrix

\[
P = \begin{pmatrix}
(T - 1)^{-1} k^1_s & 0 \\
0 & -T^{-1} k^1_s
\end{pmatrix},
\]

and compute the pseudo-difference operator

\[
AP = \begin{pmatrix}
k^1_s(T + 1)(T - 1)^{-1} k^1_s & \frac{k^1_s k^2_s}{k^2_s} T k^1_s - k^1_s T k^2_s
\end{pmatrix}
\]

which is clearly anti-symmetric. We denote it by \( \mathcal{H}[k^1_s, k^2_s] \).

**Theorem 4.5.** The operator \( \mathcal{H}[k^1_s, k^2_s] \), given by (26) is a Hamiltonian operator. It forms a Hamiltonian pair with Hamiltonian operator

\[
\mathcal{H}_0[k^1_s, k^2_s] = \begin{pmatrix}
0 & k^1_s \\
-k^1_s & 0
\end{pmatrix}.
\]

**Proof.** Let us introduce the following Miura transformation

\[
p_s = \frac{k^1_s}{k^1_{s+1}}, \quad q_s = k^2_s.
\]

Its Fréchet derivative is

\[
D_{(p_s, q_s)} = \begin{pmatrix}
\frac{1}{k^1_{s+1}} - \frac{k^1_s}{k^2_{s+1}} T & 0 \\
0 & \frac{1}{k^2_{s+1}}
\end{pmatrix}
= \begin{pmatrix}
\frac{1}{k^2_{s+1}} - p_s T & 0 \\
0 & 1
\end{pmatrix}.
\]

Under this transformation the operators \( \mathcal{H}_0[k^1_s, k^2_s] \) and \( \mathcal{H}[k^1_s, k^2_s] \) become

\[
\mathcal{H}_0[p_s, q_s] = D_{(p_s, q_s)} \mathcal{H}_0[k^1_s, k^2_s] D^*_{(p_s, q_s)} = \begin{pmatrix}
0 & p_s(1 - T) \\
-(1 - T^{-1}) p_s & 0
\end{pmatrix}
\]

and

\[
\mathcal{H}[p_s, q_s] = D_{(p_s, q_s)} \mathcal{H}[k^1_s, k^2_s] D^*_{(p_s, q_s)}
\]

\[
= \left( \begin{array}{cc}
\frac{1}{k^1_{s+1}} - p_s T & \frac{1}{k^2_{s+1}} \\
0 & 1 \end{array} \right) \left( \begin{array}{cc}
k^1_s(T + 1)(T - 1)^{-1} k^1_s & k^1_s q_s \\
-k^2_s q_s & T^{-1} p_s - p_s T
\end{array} \right) \left( \begin{array}{cc}
\frac{1}{k^1_{s+1}} - \frac{1}{k^2_{s+1}} T^{-1} p_s & 0 \\
0 & 1
\end{array} \right)
= \left( \begin{array}{cc}
p_s T^{-1} p_s - p_s T & p_s(1 - T) q_s \\
-q_s(1 - T^{-1}) p_s & T^{-1} p_s - p_s T
\end{array} \right).
\]

These two operators form a Hamiltonian pair for the well-known Toda lattice (2) in Flaschka coordinates [1, 44]. Indeed, we have

\[
\begin{pmatrix}
p(t) \\
q(t)
\end{pmatrix} = \begin{pmatrix}
p_s(q_s - q_{s+1}) \\
p_s(1 - T) q_s
\end{pmatrix} = \mathcal{H}_0 q_s = \mathcal{H}_0 \delta \left( \frac{1}{2} q^2_s + p_s \right)
\]

and thus we proved the statement.
If we take \((v^2_s, v^1_s) = (0, -k^{1}_{s-1})\) in (23), then the evolution of the invariants for \(k^1_s\) and \(k^2_s\) becomes
\[
\begin{pmatrix}
(k^1_0)\cr(k^2_0)
\end{pmatrix}_{t} = \frac{k^{1}_{s-1}}{k^{2}_{s}} \begin{pmatrix}
k^{1}_{s-1} & k^{2}_{s} \\
k^{1}_{s+1} & k^{2}_{s+1}
\end{pmatrix} = \mathcal{H} \delta \left( \frac{(k^2_s)^2}{2} + \frac{k^1_s}{k^{1}_{s+1}} \right) = \mathcal{H} \delta k^2_s,
\]
which is an integrable differential difference equation and can be transformed into Toda lattice (28) under the transformation (27). We summarize our results as a Theorem.

**Theorem 4.6.** The evolution of polygons in the centro-affine plane described by the equation
\[
(x_s)_t = \frac{k^{1}_{s-1}}{|x_s, x_{s+1}|} x_{s+1}
\]
induces a completely integrable system in its curvatures \(k^1_s, k^2_s\) equivalent to the Toda Lattice.

The pseudo-difference operator \(\mathcal{P}\) is really a formal operator as \(T - 1\) is not invertible in the periodic case. It means that we apply the operator only to Hamiltonians whose gradients are in the image of \(T - 1\), as is the case here.

**Example 4.7.**

In the projective example (see Example 3.8) we had two possible moving frames. For convenience we will choose the second one defining the invariant \(k_r\) in (11).

The subgroup \(H\) is the isotropy subgroup of 0, that is, \(H\) is given by lower triangular matrices. Thus, a section can be chosen to be
\[
\varsigma(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}.
\]

One can check directly that \(\varsigma(x)^{-1} \rho_r \in H\) and also
\[
d\varsigma(0)\nu = \begin{pmatrix} 0 & \nu \\ 0 & 0 \end{pmatrix}.
\]

A general invariant evolution in this case would be of the form
\[
(x_s)_t = d\Phi_{\rho_s}(0)\nu_s = \frac{1}{\alpha_s^2} \nu_s
\]
where \(\nu_s\) is a function of \((k_r)\) and where \(\alpha_s\) is the \((2, 2)\) entry of \(\rho_s\). The evolution induced on \(k_s\) is given by the equation (18) with
\[
N_s = \begin{pmatrix} \alpha_s & \nu_s \\ \beta_s & \alpha_s \end{pmatrix}.
\]

We have
\[
\begin{pmatrix} (k_s)_t \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} k_s & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \alpha_{s+1} & \nu_{s+1} \\ \beta_{s+1} & \alpha_{s+1} \end{pmatrix} - \begin{pmatrix} \alpha_s & \nu_s \\ \beta_s & \alpha_s \end{pmatrix} \begin{pmatrix} k_s & 1 \\ -1 & 0 \end{pmatrix}.
\]

The entries \((2, 2)\) and \((1, 2)\) of this system will allow us to solve for \(\beta_r\) and \(\alpha_r\). Indeed, the \((2, 2)\) entry is given by
\[
0 = -\beta_s - \nu_{s+1}
\]
and so
\[
\beta_s = -\nu_{s+1}.
\]
The \((1, 2)\) entry is given by
\[
\alpha_{s+1} + \alpha_s - k_s v_{s+1} = 0.
\]
This leads to
\[
\alpha_s = (T + 1)^{-1} k_s v_{s+1},
\]
where \(T\) is the shift operator. Notice that \(T + 1\) is an invertible operator. Assuming \(N\) is not even we can solve the equation \((T + 1)\alpha_s = w_s\) to obtain
\[
\alpha_s = \frac{(-1)^s}{2} \left( \sum_{i=0}^{s-1} (-1)^{i+1} w_i - \sum_{i=s}^{n-1} (-1)^{i+1} w_i \right).
\]
Finally, the \((1, 1)\) entry gives the evolution for \(k_s\)
\[
(k_s)_t = v_s - v_{s+2} + k_s (\alpha_{s+1} - \alpha_s) = (T^{-1} - T + k_s (T - 1)(T + 1)^{-1} k_s) v_{s+1}.
\]

**Theorem 4.8.** The anti-symmetric operators
\[
\mathcal{H}_1[k_s] = T - T^{-1}
\]
and
\[
\mathcal{H}_2[k_s] = k_s(T - 1)(T + 1)^{-1} k_s
\]
form a Hamiltonian pair.

**Proof.** Let us introduce the following Miura transformation
\[
u_s = \frac{1}{k_s}.
\]
We compute its Fréchet derivative \(D_{k_s} = -1/u_s^2\). Under the transformation (31) the operators \(\mathcal{H}_1[k_s]\) and \(\mathcal{H}_2[k_s]\) become
\[
\mathcal{H}_1[u_s] = D_{k_s}^{-1} \mathcal{H}_1[k_s] D_{k_s}^{-1} = u_s^2 (T - T^{-1}) u_s^2;
\]
\[
\mathcal{H}_2[u_s] = D_{k_s}^{-1} \mathcal{H}_2[k_s] D_{k_s}^{-1} = u_s(T - 1)(T + 1)^{-1} u_s.
\]
These two operators form a Hamiltonian pair for the well-known modified Volterra lattice (3) [27].
\[
(u_s)_t = u_s^2 (u_{s+1} - u_{s-1}) = \tilde{\mathcal{H}}_1[u_s] \delta_{u_s} \ln u_s = \tilde{\mathcal{H}}_2[u_s] \delta_{u_s} (u_s u_{s-1}).
\]
This implies that \(\mathcal{H}_1[k_s]\) and \(\mathcal{H}_2[k_s]\) form a Hamiltonian pair and the statement is proved. \(\square\)

Take \(v_{s+1} = \delta_{k_s} \ln k_s\) in (30). The evolution for \(k_s\) becomes
\[
(k_s)_t = \frac{1}{k_{s-1}} - \frac{1}{k_{s+1}},
\]
which is an integrable difference equation. Under the transformation (31), it leads to the modified Volterra lattice (32).

At this moment it seems natural to wonder if one could also obtain this modified Volterra evolution in the centroaffine case (4.4), via a reduction to the space \(k_s^1 = 1\) for all \(s\). And indeed such is the case.

Assume \(k_s^1 = 1\) and choose those evolutions that leave \(k_s^1\) invariant. That is, assume \((k_s^1)_t = 0\). Then, from (23) we have
\[
v_{s+1}^2 + v_s^2 - k_s^2 v_{s+1} = 0
\]
and from here
\[ v_s^2 = (T + 1)^{-1}k_s^2v_{s+1}^1. \]
Substituting in (23) we get
\[
(k_s^2)_t = -k_s^2(T + 1)^{-1}k_s^2Tv_s^1 - v_s^1 + k_s^2(T + 1)^{-1}k_s^2Tv_s^1 + T^2v_s^1 \\
= [T - T^{-1} - k_s^2(T - 1)(T + 1)^{-1}k_s^2]Tv_s^1,
\]
which is identical to the projective one with identifications \( k_s = k_s^2 \) and \( v_s = -v_s^1 \). Hence, choosing \( Tv_s^1 = -\delta k_s \ln|k_s^2| \) will result in a modified Volterra equation for the centro-affine invariant \( k_s^2 \), as far as the initial polygon satisfy \( k_s^1 = 1 \), that is, as far as the area of the parallelogram formed by \( x_s \) and \( x_{s+1} \) in the plane is equal 1 for all \( s \) and they are properly oriented. Other constant values can be chosen with minimal changes. Notice that this evolution is well-defined in the periodic case.

We will next prove the analogous to Theorem 4.3 for invariant maps. The proof is more direct.

**Theorem 4.9.** Assume \( T \) is an invariant map given as in (17). Then the map induced on the invariants is given by
\[
T(K_s) = M_s^{-1}K_sM_{s+1}
\]
where \( M_s = \rho_s^{-1}T(\rho_s) \) and where by \( T(\rho_s) \) we mean \( T(\rho_s((x_s))) = T((x_t)) \).
Furthermore, if \( \zeta \) is a section as before, and if \( \rho_s = \zeta(x_s)\rho^H_s \) for some \( \rho^H_s \in H \), then \( M_s = \zeta(z_s)M^H_s \) where \( M^H_s \in H \).

**Proof.** First of all notice that \( T(\rho^{-1}_s)T(\rho_s) = T(\rho^{-1}_s\rho_s) = T(I) = I \) and so \( T(\rho^{-1}_s) = T(\rho_s)^{-1} \). From here
\[
T(K_s) = T(\rho^{-1}_s)T(K_{s+1}) = T(\rho_s)^{-1}\rho_s\rho^{-1}_s\rho_{s+1}\rho^{-1}_{s+1}T(\rho_{s+1}) = M^{-1}_sK_sM_{s+1}.
\]
Also, assuming that \( \rho_s = \zeta(x_s)\rho^H_s \), a direct calculation using (20) shows
\[
M_s = \rho^{-1}_sT(\zeta(x_s)\rho^H_s) = \rho^{-1}_s\zeta(T(x_s))T(\rho^H_s) = \rho^{-1}_s\zeta(\rho_s \cdot z_s)T(\rho^H_s) \\
= \rho^{-1}_s\zeta(z_s)h(z_s, \rho_s)T(\rho^H_s) = \zeta(z_s)h(z_s, \rho_s)T(\rho^H_s).
\]
It suffices to call \( M^H_s = h(z_s, \rho_s)T(\rho^H_s) \in H \) to conclude the theorem. \( \square \)

**Example 4.10.**

In the centro-affine case an invariant map is of the form
\[
T(x_s) = -\frac{z_s^1}{|x_s, x_{s+1}|}x_{s+1} + z_s^2x_s = \rho^{-1}_s\begin{pmatrix} z_s^1 \\ z_s^2 \end{pmatrix}.
\]
Although in this simple case we could directly find the transformation of the invariants, we will follow the process described before as illustration. Our next example will provide a stronger case for the effectiveness of the method. Using section (22), the transformation under \( T \) of \( k_s^1 \) and \( k_s^2 \) will be built up using the matrix \( M_s = -\rho_{s+1}\rho^{-1}_s \) given by
\[
M_s = \begin{pmatrix} (z_s^2)^{-1} & z_s^1 \\ 0 & z_s^2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \alpha_s & 1 \end{pmatrix}.
\]
where \( \alpha_s \) needs to be found. The equations relating \( K_s \) and \( M_s \) are given by

\[
T \left( \frac{k^2_s}{(k^1_s)^{-1}} \begin{pmatrix} -k^1_s \\ 0 \end{pmatrix} \right) = \begin{pmatrix} 1 & 0 \\ -\alpha_s & 1 \end{pmatrix} \left( \begin{pmatrix} z^1_s \\ (z^2_s)^{-1} \end{pmatrix} \left( \frac{k^2_s}{(k^1_s)^{-1}} \begin{pmatrix} -k^1_s \\ 0 \end{pmatrix} \right) \left( \frac{z^1_{s+1}}{z^2_{s+1}} \right) \right) \left( \begin{pmatrix} 1 \\ \alpha_{s+1} \\ 1 \end{pmatrix} \right)
\]

The (2,2) entry of this equation will allow us to solve for \( \alpha_s \). It is given by

\[
\alpha_s \left( \frac{1}{k^1_s} z^1_s z^1_{s+1} + z^2_s (z^2_{s+1} k^1_s - z^1_{s+1} k^2_s) \right) + \frac{z^1_{s+1}}{z^2_s k^2_s} = 0.
\]

The other entries of the system will solve for the transformation of \( k^1_s \) and \( k^2_s \). They are given by

\[
T(k^1_s) = \frac{1}{k^1_s} z^1_s z^1_{s+1} + (z^2_{s+1} k^1_s - z^1_{s+1} k^2_s) z^2_s
\]

\[
T(k^2_s) = \left( \frac{k^2_s}{k^1_s} z^1_s z^1_{s+1} - z^2_{s+1} k^1_s z^2_s - \frac{1}{k^1_s} z^1_{s+1} z^2_{s+1} \right) \alpha_{s+1} - \frac{z^1_s}{z^2_s k^1_s} + \frac{z^2_s k^2_s}{z^2_{s+1}}
\]

Now we take \( z^2_s = \frac{1}{c} \), where \( c \neq 0 \) is constant and \( z^1_s \) satisfying the relation

\[
\frac{1}{k^1_s} z^1_s z^1_{s+1} + \frac{k^1_s}{c^2} - \frac{z^1_{s+1} k^2_s}{c} = z^1_{s+1}
\]

Then the above maps become

\[
T(k^1_s) = z^1_{s+1}
\]

\[
T(k^2_s) = c \left( \frac{z^1_{s+1}}{k^1_{s+1}} - \frac{z^1_s}{k^1_s} \right) + k^2_s
\]

Let \( a_s = \frac{k^1_s}{k^1_{s+1}} \), \( b_s = k^2_s \) and \( \beta_s = \frac{k^1_s}{k^1_{s+1}} \). Then using (35) and (36) we have

\[
T(a_s) = \frac{T(k^1_s)}{T(k^2_{s+1})} = \frac{k^1_s \beta_{s+1}}{\beta_s} = a_s \beta_{s+1} \beta_s
\]

\[
T(b_s) = b_s + c \left( \frac{k^1_s}{k^2_{s+1} \beta_{s+1}} - \frac{k^1_s}{k^1_s \beta_{s-1}} \right) = b_s + c \left( \frac{a_s}{\beta_s} - \frac{a_{s-1}}{\beta_{s-1}} \right)
\]

The constraint (34) on \( z^1_s \) becomes

\[
\frac{a_{s-1}}{\beta_{s-1} \beta_s} + \frac{1}{c^2} - \frac{b_s}{c \beta_s} = \frac{1}{\beta_s},
\]

that is,

\[
\beta_s = 1 + c b_s - c^2 \frac{a_{s-1}}{\beta_{s-1}}.
\]

Thus we obtain the integrable discretization of the Toda lattice as the formulas (3.8.2) and (3.8.3) in [44].
5. The homogenous sphere $\text{SO}(3)/\text{SO}(2)$

In this section we consider invariant evolutions on the homogeneous sphere, $\text{SO}(3)/\text{SO}(2)$. We first consider a local section using which we can describe our calculations of a discrete frame, the associated discrete Maurer-Cartan matrices, and invaraintizations. We then show that, for a certain choice of polygon evolution the resulting curvature flow is integrable, of Volterra type.

If $G = \text{SO}(3)$ and $H = \text{SO}(2)$, we consider the following splitting of the Lie algebra into subspaces $\mathfrak{so}(3) = \mathfrak{m} \oplus \mathfrak{h}$ with

\begin{equation}
\begin{pmatrix}
0 & y \\
-y^T & 0
\end{pmatrix} \in \mathfrak{m} \quad \begin{pmatrix}
A & 0 \\
0 & 0
\end{pmatrix} \in \mathfrak{h}
\end{equation}

where $y \in \mathbb{R}^n$ and $A \in \mathfrak{so}(2)$. Associated to this splitting we have a local factorization in the group into factors belonging to $H = \text{SO}(2)$ and $\exp(\mathfrak{m})$. This factorization is given by

\begin{equation}
g = g(\Theta, y) = \begin{pmatrix}
\Theta & 0 \\
0 & 1
\end{pmatrix} \begin{pmatrix}
I + \cos y y^T & \sin y \\
-\sin y y^T & \cos(\|y\|)
\end{pmatrix}
\end{equation}

where $\cos y = \frac{\cos(\|y\|) - 1}{\|y\|^2}$, $\sin y = \frac{\sin(\|y\|)}{|y|}$ and $\|y\|^2 = y^T y$. The factorization exists locally in a neighborhood of the identity.

Let $\varsigma : S^2 \to \text{SO}(3)$ be the section defined by the exponential, that is

$$
\varsigma(x) = \exp \begin{pmatrix}
0 & x \\
-x^T & 0
\end{pmatrix} = \begin{pmatrix}
I + \cos x x^T & \sin x \\
-\sin x x^T & \cos(\|x\|)
\end{pmatrix},
$$

where we are using local coordinates around the south pole (whose coordinates are zero). One clearly has that $d\varsigma(o)v = \begin{pmatrix}0 & v \\
-x^T & 0\end{pmatrix}$. The action of $\text{SO}(3)$ on the sphere associated to this section, let’s denote it by $g \cdot x$, is determined by the relation

$$
g \varsigma(x) = \varsigma(g \cdot x) h
$$

for some $h \in \text{SO}(2)$ which is also determined by this relation. Let $g$ be as in (38). Straightforward calculations show that, if $\eta = g \cdot x$, then

\begin{equation}
\sin y \eta = \sin x \Theta x + (\cos y \sin x y^T x + \sin y \cos(\|x\|)) \Theta y
\end{equation}

and

\begin{equation}
\cos(\|\eta\|) = \cos(\|y\|) \cos(\|x\|) - \sin x \sin y y^T x.
\end{equation}

The last equation can be expressed in terms of the cosine of a certain angle using the spherical law of cosines, as we will see later. To have a better idea of what the coordinates given by this section are, recall the standard geometric identification of $\text{SO}(3)/\text{SO}(2)$ with the sphere:

Given an element in $\text{SO}(3)$ we can identify the last column with a point on the sphere and the first two columns as vectors tangent to the sphere at the point. The element of $\text{SO}(2)$ acts on the two tangent vectors. With this identification, our coordinates result on

$$
p = \begin{pmatrix}
\sin x \frac{x}{\|x\|} \\
\cos(\|x\|)
\end{pmatrix}$$

being the point on the sphere. If we consider $\theta$ and $\phi$ to be the standard spherical angles, then $|x| = \phi$ and
\[
\begin{pmatrix}
\cos \theta \\
\sin \theta
\end{pmatrix}
\]. Therefore, the coordinates $x$ describe the projection of $p$ on the $xy$ plane, multiplied by the angle $\phi$. See the picture below. This might seem as a cumbersome choice, but the advantages in calculations will be worthy, plus our Serret-Frenet equations will look very familiar to the reader.

![Fig. 1](image-url)

5.1. **Moving frame and invariants.** Next we will determine a moving frame using the normalization constants $c_r^r = 0$ and $c_{r+1}^r = a_r e_1$, where $a_r$ is an invariant still to be identified. Assume $\rho_r = (\Theta_r, y_r)$ as in (38). The equation
\[
\rho_r \cdot x_r = 0
\]
determines the choice $y_r = -x_r$, and $\rho_r \cdot x_{r+1}$ is determined by the equation $\rho_r \cdot x_{r+1} = a_r e_1$ implicitly written as
\[
(41)
\sin a_r e_1 = \Theta_r (\sin x_{r+1}, x_{r+1} + (\cos x_r, \sin x_{r+1} x^T_{r+1} - \sin x_r, \cos(|x_{r+1}|)) x_r).
\]
The invariant $a_r$ is determined by the condition $\Theta_r \in SO(2)$, while this equation determines $\Theta_r$. If we impose the further condition $a_r > 0$ for all $r$, then $\Theta_r$ is uniquely determined (it might belong to different connected components of $SO(2)$ for different $r$’s).

**Theorem 5.1.** The right Maurer-Cartan matrices associated to $\rho_r$ as above are given by
\[
(42)
K_r = \exp\left(\begin{pmatrix} 0 & k_r & 0 \\
-k_r & 0 & 0 \\
0 & 0 & 0 \end{pmatrix}\right) \exp\left(\begin{pmatrix} 0 & 0 & -a_r \\
a_r & 0 & 0 \\
0 & 0 & 0 \end{pmatrix}\right).
\]
We call $a_r$ the discrete spherical arc-length invariant, and $k_r$ the discrete spherical curvature of the polygon.

**Proof.** Let us call $K_r = K_r(K_0, k_1)$ as in (38). To prove the theorem we can use the recursion equations (12) for $K_r$ to determine it. For a right matrix the equations are given by $K_r \cdot c_{r+1}^r = c_{r+1}^r$, which, when substituted in (39), and after minor simplifications, becomes
\[
0 = \sin_{a_r} e_1 + (\cos_{k_1} \sin_{a_r} a_r k_1^T e_1 + \sin_{k_1} \cos a_r) k_1.
\]
Let us re-write this as $0 = \sin a_r e_1 + X_r k_1$ so that $k_1 = -\frac{\sin a_r}{X_r} e_1$. We can in fact use the expression for $X_r$ to solve for it. Indeed, again after minor simplifications we obtain

$$X_r = \cos k_1 \sin a_r k_1^0 e_1 + \sin k_1 \cos a_r = X_r \left(1 - \cos \left(\frac{\sin a_r}{X_r}\right) + \cos a_r \sin \left(\frac{\sin a_r}{X_r}\right)\right).$$

This results in the relation

$$\sin \left(\frac{\sin a_r}{X_r}\right) \cos a_r - \cos \left(\frac{\sin a_r}{X_r}\right) \sin a_r = \sin \left(\frac{\sin a_r}{X_r} - a_r\right) = 0$$

which results in $X_r = \sin a_r$ and $k_1 = -a_r e_1$.

This calculation determines the $m$ component of $K_r$ as in the statement of the theorem. Notice that the $h$ component depends on only one parameter and therefore we can write it as in the statement, even if we do not give its explicit formula. □

Both invariants $a_r$ and $k_r$ have a very simple geometric description, as shown in our next theorem.

**Theorem 5.2.** Let $\beta_{r,s}$ be the counterclockwise angle formed by the arc $x_r x_s$ and the arc $x_s N$, where $N$ is the north pole. Then, the spherical arc-length invariant $a_r$ is the length of the arc joining $x_r$ to $x_{r+1}$ and

$$k_{r-1} = \pi - (\beta_{r,r+1} + \beta_{r+1,r+2})$$


\[\text{Fig. 2}\]

**Proof.** Calculating the length of the vector in equation (41) we see that

$$\sin^2(a_r) = \sin \left|x_{r+1}\right|^2 + \left(\cos \left|x_r\right| - 1\right) \sin \left|x_{r+1}\right| \cos \theta_{r,r+1} - \sin \left|x_r\right| \cos \left|x_{r+1}\right| \right)^2$$

where $\theta_{r,r+1} = \theta_r - \theta_{r+1}$ and $\theta_r$ is the polar angle of $x_r$. After some simplifications this equality becomes

$$\sin^2(a_r) = 1 - \left(\cos \left|x_r\right| \cos \left|x_{r+1}\right| + \sin \left|x_r\right| \sin \left|x_{r+1}\right| \cos \theta_{r,r+1}\right)^2.$$  

Using the spherical law of cosines (see figure 3) we obtain that

$$\cos \left|x_r\right| \cos \left|x_{r+1}\right| + \sin \left|x_r\right| \sin \left|x_{r+1}\right| \cos \theta_{r,r+1} = \cos \alpha_{r,r+1},$$

where $\alpha_{r,r+1}$ is the angle between $p_r$ and $p_{r+1}$, the points on the sphere corresponding to $x_r$ and $x_{r+1}$. Using this fact, the above becomes

$$-\cos^2 \alpha_{r,r+1} + 1 = \sin^2 \alpha_{r,r+1} = \sin^2 a_r.$$
Therefore, \( a_r = |\alpha_{r,r+1}| \) since it is a positive number, as stated.

Law of cosines:
\[
\cos c = \cos b \cos a + \sin b \sin a \cos C
\]

Law of sines:
\[
\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}
\]

Fig. 3

A similar calculation, although a more involved one, describes \( k_r \). Using the recursion relation
\[
K_{r} \cdot c_{r+1} = c_{r+2}^{+2},
\]
the fact that \( c_{r+1}^{+2} = 0 \), the expression of \( K_r \) and the formula (39) for the action, we have that
\[
\sin_{r+2} c_{r+1}^{+2} = \sin a_r \left( -\cos k_r \sin k_r \right).
\]

Therefore, \( \pi - k_r \) is the polar angle of \( c_{r+2}^{+2} \) (while \( |c_{r+2}^{+2}| = a_r \)). Now, we do know that \( \rho_{r+2} \cdot x_{r+1} = c_{r+1}^{+2} \), where \( \rho \) is the right moving frame; therefore, we can again use (39) to find
\[
\sin_{r+2} c_{r+1}^{+2} = \Theta_{r+2} V_{r+1,r+2}
\]
where
\[
V_{i,j} = \sin x_i x_j + (\cos x_i \sin x_i x_j \cdot x_i - \sin x_i \cos |x_i|) x_j.
\]

We clearly see that the polar angle of \( c_{r+2}^{+2} \) is a combination of the polar angle of \( \Theta_{r+2} \) and the one of \( V_{r+1,r+2} \). Now, we know that \( \Theta_r \) is determined by the equation (41). That is, \( \Theta_{r+2} \) rotates \( V_{r+3,r+2} \) to the \( x \)-axis
\[
\Theta_{r+2} (\sin x_{r+3} x_{r+3} + (\cos x_{r+3} \sin x_{r+3} x_{r+2} \cdot x_{r+3} - \sin x_{r+2} \cos \|x_{r+3}\|) x_{r+2})
= \Theta_{r+2} V_{r+3,r+2} = \sin a_{r+2} \epsilon_1 a_{r+1} \epsilon_1.
\]

If we denote by \( P(v) \) the polar angle of the vector \( v \), from here we get that \( \pi - k_r = P(V_{r+1,r+2}) - P(V_{r+3,r+2}) \).

Next, we can write \( x_r \) in terms of the polar coordinates of \( p_r \), that is,
\[
x_r = \begin{pmatrix}
\sin |x_r| \cos \theta_r \\
\sin |x_r| \sin \theta_r
\end{pmatrix}
\]
and substitute in the expression for \( V_{i,j} \). After some minor trigonometric manipulations we obtain
\[
V_{i,j} = \begin{pmatrix}
\cos \theta_j & -\sin \theta_j \\
\sin \theta_j & \cos \theta_j
\end{pmatrix}
\begin{pmatrix}
\cos \|x_j\| \sin |x_i| \cos(\theta_{i,j}) - \sin |x_j| \cos |x_i| \\
\sin \|x_j\| \sin(\theta_{i,j})
\end{pmatrix}
\]
Form here \(P(V_{r+1,r+2}) = P(V_{r+3,r+2}) = P(v_{r+1,r+2}) = P(v_{r+3,r+2})\), where
\[
\begin{aligned}
  v_{i,j} &= \left( \cos \|x_j\| \frac{\sin \|x_j\| \cos(\theta_{i,j}) - \sin \|x_j\| \cos \|x_i\|}{\sin \|x_i\| \sin(\theta_{i,j})} \right).
\end{aligned}
\]

Notice that, as before, after basic trigonometric manipulations
\[
|v_{i,j}|^2 = 1 - (\cos(\|x_i\|) \cos(\|x_j\|) + \sin(\|x_i\|) \sin(\|x_j\|)) \cos(\theta_{i,j})^2 = 1 - \cos^2 \alpha_{i,j} = \sin^2 \alpha_{i,j}.
\]
where we have used the spherical law of cosines once more. Notice that \(0 \leq \alpha_{i,j} \leq \pi\) and so from (44) the sine of the polar angle of \(v_{i,j}\) is given by
\[
\frac{\sin \|x_j\| \sin(\theta_{i,j})}{\sin \alpha_{i,j}}.
\]

Let us assume all the angles involved are in the first two quadrants. Then, the spherical law of sines tells us that
\[
\frac{\sin \alpha_{r+1,r+2}}{\sin \theta_{r+1,r+2}} = \frac{\sin \|x_r\|}{\sin \beta_{r+1,r+2}}
\]
and so the sine of the polar angle of \(v_{r+1,r+2}\) is given by
\[
\frac{\sin \|x_{r+1}\| \sin(\theta_{r+1,r+2})}{\sin \alpha_{r+1,r+2}} = \sin \beta_{r+1,r+2}.
\]
Likewise, the polar angle of \(v_{r+3,r+2}\) is \(\beta_{r+3,r+2} = -\beta_{r+2,r+3}\). Therefore,
\[
\pi - k_r = \beta_{r+1,r+2} + \beta_{r+2,r+3}
\]
as stated. \(\square\)

5.2. **Invariantization of invariant evolutions and its associated integrable system.** We will next describe the evolution induced on \(K_r\) by an invariant evolution of polygons on the sphere. According to (15) and using our definition of moving frame, the most general form for an invariant evolution of polygons on the sphere is given by
\[
(x_s)_t = \left( \sin^{-1} x_s \left( I - \frac{x_s x_s^T}{\|x_s\|^2} \right) \frac{x_s x_s^T}{\|x_s\|^2} \right) \Theta_s^{-1} r_s
\]
for some invariant vector \(r_s = \left( \frac{r^1_s}{r^2_s} \right)\) depending on \((k_r)\) and \((a_r)\).

**Theorem 5.3.** Assume the twisted polygons \((x_r)\) are solutions of an invariant evolution of the form above. Then the invariants \(a_s\) and \(k_s\) evolve following the equations
\[
\begin{aligned}
  (a_s)_t &= -r^1_s + r^1_{s+1} \cos k_s - r^2_{s+1} \sin k_s, \\
  (k_s)_t &= \frac{1}{\sin a_s} - \frac{r^2_s}{\sin a_{s+1}} \frac{\cos a_{s+1}}{\sin a_{s+1}} + \frac{r^1_s}{\sin a_{s+1}} \frac{\sin k_{s+1}}{\sin a_{s+1}} + \frac{r^2_s}{\sin a_{s+1}} \frac{\cos k_{s+1}}{\sin a_{s+1}} \\
  &\quad - \frac{r^1_{s+1}}{\sin a_s} \frac{\cos a_s}{\sin a_{s+1}} \sin k_s - \frac{r^2_{s+1}}{\sin a_{s+1}} \frac{\cos a_s}{\sin a_s} \cos k_s.
\end{aligned}
\]

**Proof.** Assume the polygons are evolving following this evolution; then, our left Maurer-Cartan matrices \(\tilde{K}_s = K_s^{-1}\) will follow the evolution
\[
\tilde{K}_s^{-1}(\tilde{K}_s)_t = N_{s+1} - \tilde{K}_s^{-1} N_s \tilde{K}_s
\]
Next, we remark that 

\[
\begin{pmatrix}
  0 & -k_1 \\
  k_1 & 0
\end{pmatrix}
\]

commutes with its derivative since \( k_1 \) is a multiple of \( e_1 \). This means

\[
\left( \exp \left( \begin{pmatrix}
  0 & -k_1 \\
  k_1 & 0
\end{pmatrix}
\right) \right)_t = \left( \begin{pmatrix}
  0 & 0 & (a_s)_t \\
  0 & 0 & 0 \\
  -(a_s)_t & 0 & 0
\end{pmatrix}
\right) \exp \left( \begin{pmatrix}
  0 & -k_1 \\
  k_1 & 0
\end{pmatrix}
\right).
\]

Using the expression of \( K_s \) in the previous theorem, one has that

\[
\tilde{K}_s^{-1}(\tilde{K}_s)_t = \left( \begin{pmatrix}
  0 & -(k_s)_t & (a_s)_t \cos k_s \\
  (k_s)_t & 0 & -(a_s)_t \sin k_s \\
  -(a_s)_t \cos k_s & (a_s)_t \sin k_s & 0
\end{pmatrix}
\right).
\]

Substituting this into (48) and multiplying out matrices we get

\[
(k_s)_t = -n_{s+1} + n_s \cos a_s + r_s^2 \sin a_s
\]

and

\[
\begin{pmatrix}
  \cos k_s & \sin k_s \\
  -\sin k_s & \cos k_s
\end{pmatrix} \begin{pmatrix}
  (a_s)_t \\
  0
\end{pmatrix} = \begin{pmatrix}
  \cos k_s & \sin k_s \\
  -\sin k_s & \cos k_s
\end{pmatrix} \begin{pmatrix}
  -r_s^1 \\
  r_s^2 \cos a_s + n_s \sin a_s
\end{pmatrix} = \begin{pmatrix}
  r_{s+1}^1 \\
  r_{s+1}^2 \cos k_s
\end{pmatrix}.
\]

From here we get

\[
n_s = r_s^2 \frac{\cos a_s}{\sin a_s} - \frac{1}{\sin a_s} \left( r_{s+1}^1 \sin k_s + r_{s+1}^2 \cos k_s \right)
\]

and

\[
(a_s)_t = -r_s^1 - r_{s+1}^2 \sin k_s + r_s^1 \cos k_s.
\]

Substituting the value of \( n_s \) in the evolution of \( k_s \) completes the proof. \( \square \)

In particular, if we ask that the evolution preserves the spherical arc-length (i.e., \( (a_s)_t = 0 \)) and we choose the value \( \tan a_s = 1 \) as constant value, then choosing \( r_s^1 = 1 \) for all \( s \) produces the curvature evolution

\[
(k_s)_t = \sqrt{2} \left( \frac{1 - \cos k_{s+1}}{\sin k_{s+1}} - \frac{1 - \cos k_{s-1}}{\sin k_{s-1}} \right).
\]

Let \( u_s = \frac{1 - \cos k_s}{\sin k_s} \). The above equation becomes

\[
(u_s)_t = \frac{\sqrt{2}}{2} (1 + u_s^2) (u_{s+1} - u_{s-1}),
\]

which is a special case of equation (V1) in the list of integrable Volterra-type equations [47]. It is a bi-Hamiltonian equation [48], where the second Hamiltonian operator was not explicitly given. Here we give its compatible Hamiltonian and symplectic operators [4]. Equation (49) can be written as

\[
(u_s)_t = \frac{\sqrt{2}}{4} \mathcal{H} \delta_{u_s} \ln(1 + u_s^2) \quad \text{and} \quad \mathcal{I}(u_s)_t = \frac{\sqrt{2}}{2} \delta_{u_s} ((1 + u_s^2) u_{s+1} u_{s-1} + \frac{1}{2} u_s^2 u_{s-1}^2)
\]

where

\[
\mathcal{H} = (1 + u_s^2) (T - T^{-1}) (1 + u_s^2)
\]
is Hamiltonian and
\[ I = T - T^{-1} + \frac{2u_s}{1 + u_s^2}(T - 1)^{-1}(u_{s+1} + u_{s-1}) + (u_{s+1} + u_{s-1})T(T - 1)^{-1} \frac{2u_s}{1 + u_s^2} \]
is a sympletic operator. Furthermore, these two operators satisfy
\[ \mathcal{H}I = \mathcal{R}^2 = \left( (1 + u_s^2)T + 2u_su_{s+1} + (1 + u_s^2)T^{-1} + (u_s)(T - 1)^{-1} \frac{2u_s}{1 + u_s^2} \right)^2, \]

where \( \mathcal{R} \) is a Nijenhuis recursion operator of (49). Its recursion operator \( \mathcal{R} \) can not be written as the product of weakly nonlocal Hamiltonian and symplectic operators. A similar example was presented in [46].

Notice that, as before, \( T - 1 \) is not invertible in the periodic case. That means we will need to work on infinite gons, or assume we are working with Hamiltonians whose gradient is in the image of the operator, as it is the case with this example.

6. CONCLUSION AND FUTURE WORK

In this paper, we have developed the notion of a discrete moving frame and shown it has computational advantages over a single frame for the main application we consider, which is the invariatization of discrete evolution flows and mappings. Further, we have shown that in the examples we consider, the use of discrete moving frames greatly aids the identification of discrete integrable systems and biPoisson maps that can be obtained as invariatizations of invariants flow of gons.

Investigations are under way on how the Hamiltonian structures might appear in the general case, and in particular how they appear in the projective plane. Further work also remains to illustrate and detail how our constructions of invariant mappings and their invariantizations lead in general to discrete integrable mappings, as seen in Example 4.10. This work is different in nature to the previous one and it will be greatly aided by understanding the differential-difference problem.

Hamiltonian structures in the projective plane could be relevant to the results in [39]. There the authors proved that the so-called pentagram map is completely integrable, but they did not provide a biHamiltonian structure for its invariatization. A structure obtained in this setting would be a natural candidate to prove that the pentagram map is biPoisson.

Other applications of discrete moving frames remaining to be developed include applications to the discrete Calculus of Variations, as briefly indicated in Example 2.6. We believe however, that the applications will extend to invariant numerical schemes and computer graphics, amongst others.

REFERENCES


