Topics covered on this exam

The first midterm will be on the material covered in sections 1–16 of the lecture notes.

- **Integration.** You should be able to use the following methods to compute integrals.
  - Substitution
  - Integration by parts
  - Reduction formulas: know how to use them if you are given one, and know how to derive one.
  - Partial fractions.
- **Taylor polynomials.**
  - You should know how to compute $T_\infty^a f(x)$ (i.e. give a formula for the coefficient of x^n) by computing $f^{(n)}(a)$.
  - Use partial fractions to compute Taylor series of simple rational functions.
  - Use Lagrange's formula for the remainder to estimate the error term $R_n f(x)$ if x and n are explicitly given, or "for $|x| \leq 0.2$".
  - Understand the "little-oh" notation.
  - Use a combination of "little-oh", "standard Taylor series", algebra of Taylor polynomials (including "$T_n[f'(x)] =$ derivative of $T_{n+1}[f(x)]$") to efficiently compute Taylor polynomials.

Theory Questions

The following questions may not necessarily appear on the midterm, but answering them will help you find gaps in your understanding of the material.

(1) State the Fundamental Theorem of Calculus.
(2) The integral $\int_0^1 4(x+a)^3 \mathrm{d}x$ contains two variables, x and a. Which of these is a dummy variable and why?
(3) State and derive the formula for integration by parts.
(4) Explain to a friend what the partial fraction expansion of a rational function looks like and how you find it.
(5) What is the Taylor polynomial of degree n of a function $f(x)$ at the point a?
(6) State the Geometric Sum Formula.
(7) What is the difference between a Taylor and a Maclaurin polynomial?
(8) What is the remainder term in a Taylor polynomial?
(9) State Lagrange's formula for the remainder term in a Taylor polynomial.
(10) What does it mean for a function $f(x)$ to be $o(x^n)$ as $x \to 0$?

Sample Problems for the 1st Midterm

These are problems from 222 midterms that were given in the past, and they are relevant to the coming first midterm. Other old exams are available on the web, but be warned that the syllabus changes every year, so that problems which were on the 1st midterm 2 years ago may show up on this year's 2nd midterm, and *vice versa*. Don't prepare for last year's midterm!

(1) Compute the following integral, in which a may be any constant.

$$\int e^x \cos ax \, dx.$$

(2) Find a reduction formula for

$$I_n = \int_0^1 \bigl(1+x\bigr)^n e^{-x} \, dx,$$

and compute I_3.

(3) (a) Find the partial fraction decomposition of $\dfrac{x^2}{x^2-3x+2}$.

(a) Find the partial fraction decomposition of $\dfrac{x^2-3x+2}{(x^2+1)(x-4)^3}$. You do not have to find the unknown coefficients "A,B, $\cdots$"

(4) Compute $\int_1^2 (6x^2-2x)\ln(x)\,dx$.

Then compute $\int_1^2 (6x^2-2x)\ln(x^2)\,dx$.

(5) Find the term containing x^{13} in the Taylor series $T_\infty f(x)$ of $f(x)=\cos(x+\pi/6)$.

(6) Let $f(x)=\ln(5+x)$. Compute the Taylor series $T_\infty f(x)$.

(7) Compute the Taylor series $T_\infty[f(x)]$ of $f(x)=\dfrac{3-x}{2-3x+x^2}$.

(8) (a) Find the Taylor series of $f(x)=\dfrac{2x}{1+x^2}$.

(b) Find the Taylor series of $f'(x)$.

(9) **(2004, M1)**

(a) Find the partial fraction decomposition of $\dfrac{x+2}{x^3-x}$.

(b) Find the partial fraction decomposition of $\dfrac{x^2+1}{(x^2+4)^2(x^2-4)^3}$. You do not have to find the unknown coefficients "A,B, $\cdots$"

(10) **(2004, M1)** Compute $\int_\alpha^{\pi/2} \cos x \ln\bigl(\sin x\bigr)\,dx$, where α is some constant satisfying $0<\alpha<\pi$.

(11) **(2004, M1)** Find a reduction formula for

$$I_n = \int \bigl(\ln x\bigr)^n \mathrm{d}x$$

(12) **(2004, M1)**

(a) Compute $T_n f(x)$ where $f(x)=\ln(2+x)$.

(b) Estimate the error you make when you approximate $f(x)=\ln(2+x)$ by $T_2 f(x)$, assuming that $|x|\le 0.1$.

(13) **(2004, M1)**

(a) Find the partial fraction decomposition of $\dfrac{x-3}{x^3-x}$.

(b) Find the partial fraction decomposition of $\dfrac{x^2+1}{(x^2+9)^2(x^2-25)^3}$. You do not have to find the unknown coefficients "A, B, $\cdots$"

(14) **(2004, M1)** Compute $\int_0^\alpha \sin x \ln\bigl(\cos x\bigr)\,dx$, where α is some constant satisfying $0<\alpha<\pi$.

(15) **(2004, M1)** Find a reduction formula for

$$I_k = \int \bigl(\ln x\bigr)^k \mathrm{d}x$$

(16) **(2004, M1)**

(a) Compute $T_n f(x)$ where $f(x)=\ln(3+x)$.

(b) Estimate the error you make when you approximate $f(x)=\ln(3+x)$ by $T_2 f(x)$, assuming that $|x|\le 0.1$.

(17) **(2004, M2)** Compute $\lim_{x\to 0} \dfrac{e^{x^2}\cos x - 1 - \frac{1}{2}x^2}{1-\cos(x^2)}$

[Hint: For partial credit first compute the Taylor expansions up to $o(x^4)$ of $e^{x^2}\cos x$ and of $\cos(x^2)$.]

(18) **(2006, M2)**

(a) Compute the Taylor expansion up to $o(x^7)$ of $f(x)=\ln(1+x^3)\;-\;x\sin x^2$.

(b) Compute the following limits

$$A = \lim_{x\to 0} \frac{f(x)}{x^5}, \quad B = \lim_{x\to 0} \frac{f(x)}{x^6}, \quad C = \lim_{x\to 0} \frac{f(x)}{x^7}.$$

where $f(x)$ is as above.

(19) **(2008, M2)** Show that $\dfrac{2-x}{2+x} = e^{-x} + o(x^2)$.

(20) **(2008, M2')** Show that $\dfrac{2+x}{2-x} = e^{x} + o(x^2)$.

Answers and comments on some of the theory questions

For most of these questions you should read the notes.

1 A 221 question really, but it's the reason we want to know integrals. The FTC says that if $F'(x) = f(x)$ then $\int_a^b f(x)\mathrm{d}x = F(b) - F(a)$.

2 If you compute the integral you get $(a+1)^4 - a^4$. Therefore x is a dummy variable *because if you replace x by any other variable the integral doesn't change.* On the other hand if you replace a by some other variable the integral does change (if you replace a by b the integral becomes $(b+1)^4 - b^4$.)

Note that you don't actually have to calculate the integral to reach this conclusion. In a definite integral the integration variable is always a dummy variable, while any other variable will not be a dummy variable.

3 See the notes. Remember that it comes from the product rule.

6 $1 + x + x^2 + \cdots + x^n = \frac{1-x^{n+1}}{1-x}$. Derivation: write out what you get if you multiply both sides of this equation with $1 - x$.

7 A Maclaurin polynomial is a Taylor polynomial of some function at the point $a = 0$. In a Taylor polynomial a can be any number. Any Maclaurin polynomial is a Taylor polynomial, but not the other way around.

8 This question asks you for the *definition* of the remainder term. By definition it is the difference between the function and its Taylor polynomial. In a formula: $R_n^a[f(x)] = f(x) - T_n^a[f(x)]$.

9 Joseph-Louis Lagrange (1736–1813) said (and proved) that there always is some number c between a and x such that $R_n^a[f(x)] = \dfrac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$ (provided f is an $n+1$ times differentiable function between a and x.)

Answers to some of the sample problems

1 See the notes, in particular problem 72/73.

2 Integrate by parts once to get

$$I_n = -2^n e^{-1} + 1 + nI_{n-1}.$$

For $n = 0$ you get $I_0 = 1 - 1/e$. Hence

$$I_3 = -\frac{8}{e} + 1 + 3\{-\frac{4}{e} + 1\} + 3 \cdot 2\{-\frac{2}{e} + 1\} + 3 \cdot 2 \cdot 1\{-1/e + 1\}.$$

3 $\dfrac{x^2}{x^2-3x+2} = 1 + \dfrac{3x-2}{(x-1)(x-2)} = 1 + \dfrac{-1}{x-1} + \dfrac{4}{x-2}.$

$\dfrac{x^2-3x+2}{(x^2+1)(x-4)^3} = \dfrac{Ax+B}{x^2+1} + \dfrac{C}{x-4} + \dfrac{D}{(x-4)^2} + \dfrac{E}{(x-4)^3}.$

4 $\displaystyle\int_1^2 (6x^2-2x)\ln(x)\,dx = \left[(2x^3-x^2)\ln x\right]_1^2 - \int_1^2 \frac{2x^3-x^2}{x}\mathrm{d}x = -\int_1^2 (2x^2-x)\mathrm{d}x$ Etcetera.

Then compute $\int_1^2 (6x^2-2x)\ln(x^2)\,dx$. *Don't do another integral! Note that this is twice the previous integral since* $\ln(x^2) = 2\ln x$.

5 $f^{(13)}(x) = -\sin(x+\pi/6)$ so the coefficient of x^{13} is $f^{(13)}(0)/(13!) = -\sin(\pi/6)/13! = -1/(2\cdot 13!)$.

Here you have to know that the "*coefficient*" is the constant multiplying the power x^{13}, so that $-\frac{x^{13}}{2\cdot 13!}$ is **not** the right answer.

6 For $n \geq 1$ one has $f^{(n)}(x) = (-1)^{n-1}(n-1)!(5+x)^{-n}$ so the coefficient of x^n in $T_\infty f(x)$ is $\dfrac{f^{(n)}(0)}{n!} = \dfrac{(-1)^{n-1}}{5^n n}$. Hence

$$T_\infty f(x) = \ln 5 + \frac{x}{5} - \frac{x^2}{2\cdot 5^2} + \cdots + \frac{(-1)^{n-1}x^n}{5^n n} + \cdots$$

7

$$\begin{aligned}
\frac{3-x}{2-3x+x^2} &= \frac{3-x}{(1-x)(2-x)} && \text{factor denominator}\\
&= \frac{A}{1-x} + \frac{B}{2-x} && \text{partial fractions}\\
&= \frac{2}{1-x} - \frac{1}{2-x}\\
&= \big(2-\frac{1}{2}\big) + \big(2-\frac{1}{2^2}\big)x + \cdots + \big(2-\frac{1}{2^{n+1}}\big)x^n + \cdots
\end{aligned}$$

8a The definition leads to a big headache. Instead substitute $t = -x^2$ in the Geometric Series for $1/(1-t)$ and multiply with $2x$. You get

$$T_{2n+1}[f(x)] = 2x - 2x^3 + 2x^5 - 2x^7 + \cdots + (-1)^n 2x^{2n+1}.$$

8b Since $T_{2n}[f'(x)] = \frac{\mathrm{d}}{\mathrm{d}x}T_{2n+1}[f(x)]$ you get

$$T_{2n}[f'(x)] = 2 - 6x^2 + 10x^4 - 14x^6 + \cdots + (-1)^n(4n+2)x^{2n}.$$

9a $\dfrac{x+2}{x^3-x} = \dfrac{x+2}{x(x-1)(x+1)} = \dfrac{A}{x} + \dfrac{B}{x-1} + \dfrac{C}{x+1}$ with $A = -2, B = 3/2, C = 1/2$.

9b

$$\frac{x^2+1}{(x^2+4)^2(x^2-4)^3} = \frac{x^2+1}{(x^2+4)^2(x-2)^3(x+2)^3} =$$
$$\frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{(x-2)^3} + \frac{D}{x+2} + \frac{E}{(x+2)^2} + \frac{F}{(x+2)^3} + \frac{Gx+H}{x^2+4} + \frac{Ix+J}{(x^2+4)^2}$$

10

$$\begin{aligned}\int_0^\alpha \sin x \ln\bigl(\cos x\bigr)\, dx &= -\int_1^{\cos\alpha} \ln u\, du && (u=\cos x)\\ &= -\bigl[u\ln u - u\bigr]_1^{\cos\alpha} && (\textstyle\int \text{ by parts})\\ &= \bigl[u - u\ln u\bigr]_1^{\cos\alpha}\\ &= \cos\alpha - \cos\alpha\ln\cos\alpha - 1\end{aligned}$$

11 To find a reduction formula for

$$I_k = \int \bigl(\ln x\bigr)^k \mathrm{d}x$$

see the problems in the notes (79 and on).

12a $T_n\bigl[\ln(2+x)\bigr] = \ln 2 + \dfrac{x}{2} - \dfrac{x^2}{2^2\cdot 2} + \dfrac{x^3}{2^3\cdot 3} + \cdots + (-1)^{n+1}\dfrac{x^n}{2^n\cdot n}.$

12b Lagrange says $\ln(2+x) = \ln 2 + \frac{x}{2} - \frac{x^2}{8} + R_2\bigl[\ln(2+x)\bigr]$, where the remainder is given by

$$R_2\bigl[\ln(2+x)\bigr] = \frac{-(2+c)^{-3}}{3!}x^3,\ \text{with } c \text{ between 0 and } x.$$

The quantity $(2+c)^{-3}$ gets larger when c gets smaller. Since we are told that x satisfies $-0.1 \le x \le 0.1$ and since c lies between 0 and x, the quantity $(2+c)^{-3}$ will be at most $(2-0.1)^{-3} = \dfrac{1}{1.9^3}$ (don't simplify, you don't have a calculator). Therefore we get the following estimate for the "error" or "remainder"

$$\bigl|R_2\bigl[\ln(2+x)\bigr]\bigr| \le \frac{(1.9)^{-3}}{3!}|x^3| \le \frac{0.1^3}{1.9^3\cdot 3!}$$

In practice you would want to simplify this to get a decimal number. On a midterm you can leave this as your answer.

16a If $f(x) = \ln(3+x)$ then

$$T_n f(x) = \ln 3 + \frac{x}{3\cdot 1} - \frac{x^2}{3^2\cdot 2} + \frac{x^3}{3^3\cdot 3} - \cdots + (-1)^{n+1}\frac{x^n}{3^n\cdot n}.$$

17 Follow the hint and compute:

$$\begin{aligned} e^{x^2} &= 1 + x^2 + \tfrac12 x^4 + o(x^4) && (\text{substitute } t = x^2 \text{ in } e^t)\\ \cos x &= 1 - \tfrac12 x^2 + \tfrac1{24}x^4 + o(x^4)\\ e^{x^2}\cos x &= 1 + \tfrac12 x^2 + \tfrac1{24}x^4 + o(x^4) && (\text{multiply})\\ e^{x^2}\cos x - 1 - \tfrac12 x^2 &= \tfrac1{24}x^4 + o(x^4)\\ \cos x^2 &= 1 - \tfrac12 x^4 + o(x^4) && (\text{put } t = x^2 \text{ in } \cos t)\\ 1 - \cos x^2 &= \tfrac12 x^4 + o(x^4)\end{aligned}$$

So you get

$$\frac{e^{x^2}\cos x - 1 - \frac12 x^2}{1-\cos(x^2)} = \frac{\frac1{24}x^4 + o(x^4)}{\frac12 x^4 + o(x^4)} = \frac{\frac1{24} + \frac{o(x^4)}{x^4}}{\frac12 + \frac{o(x^4)}{x^4}}$$

By definition of "little-oh" you have

$$\lim_{x\to 0}\frac{o(x^4)}{x^4} = 0$$

so that

$$\lim_{x\to 0}\frac{e^{x^2}\cos x - 1 - \frac12 x^2}{1-\cos(x^2)} = \frac{1/24}{1/2} = \frac{1}{12}$$

18a

$$\begin{aligned}
\ln(1+x^3) &= x^3 - \tfrac{1}{2}x^6 + o(x^7) && \text{(substitute } t = x^3 \text{ in } \ln(1+t)\text{)} \\
x\sin x^2 &= x^3 - \tfrac{1}{6}x^7 + o(x^7) = x^3 + o(x^7) && \text{(set } t = x^2 \text{ in } \sin t \text{ and multiply)} \\
f(x) &= -\tfrac{1}{2}x^6 + o(x^7) && \text{(subtract)}
\end{aligned}$$

18b From part (a) we find

$$\begin{aligned}
\frac{f(x)}{x^5} &= -\tfrac{1}{2}x + o(x^2) \\
\frac{f(x)}{x^6} &= -\tfrac{1}{2} + o(x) \\
\frac{f(x)}{x^7} &= -\frac{1}{2x} + o(1)
\end{aligned}$$

Therefore $A = 0$, $B = -\frac{1}{2}$ and C does not exist.

19 Compute the Taylor series of $\frac{2-x}{2+x}$ and e^{-x} to $o(x^2)$ and note that they are the same.