- PRACTICE PROBLEMS - FIRST MIDTERM MATH 375 -

FALL SEMESTER

- For which values of $c$ are the vectors $(-1,0,0), (2,1,2), (1,1,c)$ linearly independent?
  - Do the polynomials $t^2 + 1$, $t + 1$, $t^2 + t$ span $P^2$? Explain.
  - Do the matrices
    \[
    \begin{pmatrix}
    1 & 0 \\
    -1 & 1 \\
    
    \end{pmatrix}, \begin{pmatrix}
    1 & -1 \\
    1 & 1 \\
    \end{pmatrix}, \begin{pmatrix}
    1 & 0 \\
    0 & 0 \\
    \end{pmatrix}, \begin{pmatrix}
    1 & 0 \\
    -1 & -1 \\
    \end{pmatrix}
    \]
    span $M_{2\times 2}$? Explain.

- Find the dimension of the subspace of $P^2$ given by polynomials of the form $at^2 + bt + c$ with $c = 2a + b$. Find its dimension. Find a basis. Find the coordinates of $p(t) = 3t^2 - 2t + 4$ with respect to your basis.
  - Show that the subset of $\mathbb{R}^3$ given by vectors of the form $(x, 2x+y, -y)$, with $x,y \in \mathbb{R}$, is a vector subspace of $\mathbb{R}^3$. Find its dimension and a basis for it. Find the coordinates of $(1, -1, 3)$ in the basis you chose.
  - Show that set of diagonal matrices (matrices whose only non zero entries are those in place $(1,1),(2,2),\ldots$) is a subspace of the the vector space $M_{n\times n}$ for any $n$. Give a basis and its dimension.

- Consider the transformation $L : P^2 \to P^3$ given by
  \[
  L(p) = \int_0^x p(t)dt.
  \]
  Show that $L$ is a linear transformation. Is $q(x) = x^2 + 1$ in the image of $L$? Explain.
  - Consider the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by
    \[
    T(x,y,z) = (x + z, x + y, y - z).
    \]
    Find the kernel of the transformation. Find a basis for the kernel. Is the dimension of $T(\mathbb{R}^3)$ equal to 3? Explain.
  - Consider the linear transformation $T : \mathbb{R}^3, \to \mathbb{R}^3$ defined by $T(x,y,z) = (x + 2y, y + z, x + y)$. Show that $T$ is invertible. Find its inverse. Find the matrix associated to $T$ and the one associated to its inverse with respect to the standard basis $(1,0,0), (0,1,0), (0,0,1)$ in both the domain and the image of $T$.

- Consider the space of continuous functions $f : [-1,1] \to \mathbb{R}$ such that $f(-1) = 2f(1)$. Show that this space is a vector space.
  - Consider the linear transformation $T(x,y,z) = (\frac{1}{\sqrt{3}}(x + z), y, \frac{1}{\sqrt{2}}(x - z))$.
    - Find its kernel.
    - Find the angle between $(x,0,z)$ and $T(x,0,z)$.
Apply the Gram-Schmidt process to find an orthonormal basis for the subspace of \( \mathbb{R}^4 \) spanned by \(
abla \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} \). Find the projection of \( \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \) on this subspace.

Find an orthonormal basis for \( P^2 \) with the standard inner product \( \langle a + bt + ct^2, \hat{a} + \hat{b}t + \hat{c}t^2 \rangle = a\hat{a} + b\hat{b} + c\hat{c} \).

Find an orthonormal basis for \( P^2 \) over \([0, 1]\) with the inner product \( \langle p, q \rangle = \int_0^1 p(t)q(t)dt \).

Find the projection of the function \( f(x) \) on the space of trigonometric polynomials of order 1 on \([-\pi, \pi]\) (the projection is called the Fourier polynomial of degree 1), where
\[
f(x) = \begin{cases} -\pi & \text{if } x \in [-\pi, 0] \\ \pi & \text{if } x \in [0, \pi] \end{cases}
\]

Show that if \( x, y \in V \) where \( V \) is a vector space with an inner product \( \langle \cdot, \cdot \rangle \), then \( \|x + y\|^2 + \|x - y\|^2 = \|x\|^2 + \|y\|^2 \).

– Find the set of all matrices \( A \in M_{2 \times 2} \) such that \( AB = BA \) where \( B = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \)
  
  – Show that the set is a vector subspace of \( M_{2 \times 2} \).
  
  – Find its dimension and a basis.
  
  – Find \( B^{-1} \).

Suppose \( A \) is an invertible matrix and \( \alpha \) is a nonzero number. Prove that \( \alpha A \) is invertible and \( (\alpha A)^{-1} = \frac{1}{\alpha} A^{-1} \).

Determine whether the following system has a unique solution, no solution, or infinite solutions, depending on the value of \( a \).
\[
\begin{align*}
2x - 6y + 4z &= 2 \\
4x + y + 3z &= -1 \\
3x + 4y + az &= 1
\end{align*}
\]

– Find the inverse of the matrix
\[
A = \begin{pmatrix}
1 & -\frac{1}{2} & 0 \\
-\frac{1}{3} & 1 & -\frac{1}{2} \\
0 & -\frac{1}{2} & 1
\end{pmatrix}.
\]

– Solve the system of equations
\[
\begin{align*}
x - \frac{1}{2} y &= 1 \\
-\frac{1}{2} x + y - \frac{1}{2} z &= -1 \\
-\frac{1}{2} y + z &= 1
\end{align*}
\]
• Find the solution of the system of equations

\[
\begin{align*}
  x - 2y + z &= 1 \\
  x + 2z &= -2 \\
  2y + z &= -3
\end{align*}
\]

and write the solution as a solution \( p \) plus the solution of the corresponding homogeneous equation (if the solution is unique, you still need to say what the solution of the homogeneous equation will be).

• True-False, if you think the statement is true you need to prove it. If you think it is false, show a counterexample
  - If \( S \) and \( T \) are subspaces of a vector space \( V \), then so is \( S \cap T \).
  - If \( S \) and \( T \) are subspaces of a vector space \( V \), then so is \( S \cup T \).
  - The transformation \( T : \mathbb{R}^3 \to \mathbb{R}^3 \) given by \( T(x, y) = (x, y, 1) \) is a linear transformation.
  - If the map \( T : V \to W \) is 1-to-1, then the dimension of \( V \) is equal to the dimension of \( W \).
  - If \( \{v_1, \ldots, v_n\} \) is a spanning set for a \( n \)-dimensional vector space \( V \), then they form a basis.
  - If the vectors \( \{v_1, v_2, v_3\} \) are linearly independent, then the vectors \( \{w_1, w_2, w_3\} \) where \( w_1 = v_1 + v_2 + v_3 \), \( w_2 = v_2 + v_3 \) and \( w_3 = v_3 \) are also linearly independent.

• Challenge problem (for extra credit): Let \( S \subset V \) be a linear subspace of a finite dimensional vector space \( V \). Assume \( V \) has an inner product \( \langle \cdot, \cdot \rangle \). Define \( S^\perp \subset V \) to be the set

\[
S^\perp = \{ w \in V, \text{ such that, } \langle w, s \rangle = 0, \text{ for all } s \in S \}.
\]

  - Show that \( S^\perp \) is a subspace.
  - Show that, if \( \{s_1, \ldots, s_r\} \) is a basis for \( S \) and \( \{w_1, \ldots, w_k\} \) is a basis for \( S^\perp \), then \( \Gamma = \{s_1, \ldots, s_r, w_1, \ldots, w_k\} \) is an independent set.
  - What do you think \( L(\Gamma) \) is? (do not prove).