Convergence Tests

You need to know how to use all of these!!!

0.1 Preliminaries

First, we need to note the difference between a sequence and a series.

A sequence is an infinitely long list of numbers. We usually define this sequence by describing the nth term. For example $a_n = \frac{1}{2^n}$ for $n \geq 0$ defines the sequence $a_0 = 1, a_1 = \frac{1}{2}, a_2 = \frac{1}{4}, a_3 = \frac{1}{8}, \ldots$. A series can be defined for any sequnce by adding up all of the entries in that sequence. We can ask two different questions about a sequence:

1. Does the sequence converge?
2. Does the series generated by the series converge?

To answer the first question, we compute $\lim_{n\to\infty} a_n$. If this limit exists and it is finite then our sequence converges. Otherwise, it diverges.

The second question has a very similar answer, and many people confuse the two. Don't do this. To find out if our sequence converges, we compute $\lim_{N\to\infty} \sum_{n=0}^{N} a_n$. If this limit exists and it is finite then our SERIES converges. Otherwise, it diverges. The second question is actually much harder than the first. In fact, we should be able to directly answer the first question for all the sequences in this entire chapter. However, the associated series are much more complicated. To deal with this harder question, we have many different tests to use to determine the divergence or convergence of series.

For example, we should all recognize that the example sequence defines a geometric series that sums to 2. However, without the geometric series test, we would have to do the following:

$$\begin{aligned}\sum_{n=0}^{\infty} a_n &= \sum_{n=0}^{\infty} \frac{1}{2^n} \\ &= \lim_{N\to\infty} \sum_{n=0}^{N} \frac{1}{2^n} \\ &= \lim_{N\to\infty} \frac{1 - \frac{1}{2^{N+1}}}{1 - \frac{1}{2}} = \frac{1-0}{1-\frac{1}{2}} = 2.\end{aligned}$$

This simple example should show how complicated things would get with out all of the convergence tests. So, without further ado, here they are.

0.2 The Tests

1. nth-term Test for $\sum_{n=0}^{\infty} a_n$

   If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=0}^{\infty} a_n$ diverges. Note that this is a one-way implication. This test does not apply if the nth-term converges to zero.

   Examples:

   a $\sum_{n=0}^{\infty} \frac{n^4+4n+3}{\sqrt{n^8-2n^3+1}}$
   Note that the dominating term in the numerator and denominator are the same so we divide each by the dominating term to get:

$$\begin{aligned}\lim_{n\to\infty} \frac{n^4+4n+3}{\sqrt{n^8-2n^3+1}} &= \lim_{n\to\infty} \frac{(n^4+4n+3)\cdot\frac{1}{n^4}}{\sqrt{n^8-2n^3+1}\cdot\frac{1}{n^4}} \\ &= \lim_{n\to\infty} \frac{1+\frac{4}{n^3}+\frac{3}{n^4}}{\sqrt{\frac{n^8-2n^3+1}{n^8}}} \\ &= \lim_{n\to\infty} \frac{1+\frac{4}{n^3}+\frac{3}{n^4}}{\sqrt{1-\frac{2}{n^5}+\frac{1}{n^8}}} = 1\end{aligned}$$

   Hence, the series diverges.

   b $\sum_{n=0}^{\infty} \frac{n^3}{\sqrt{n^5+2}}$
   Note that the dominating term in the numerator grows faster than that in the denominator. Thus, we expect the SEQUENCE to diverge. We show this as follows:

$$\begin{aligned}\lim_{n\to\infty} \frac{n^3}{\sqrt{n^5+2}} &= \lim_{n\to\infty} \frac{n^3\cdot\frac{1}{n^{\frac{5}{2}}}}{\sqrt{n^5+2}\cdot\frac{1}{n^{\frac{5}{2}}}} \\ &= \lim_{n\to\infty} \frac{n^{\frac{1}{2}}}{\sqrt{\frac{n^5+2}{n^5}}} \\ &= \lim_{n\to\infty} \frac{n^{\frac{1}{2}}}{\sqrt{1+\frac{2}{n^5}}} = \infty\end{aligned}$$

   The SEQUENCE diverges, so in particular, the SEQUENCE does not go to zero. Thus, the SERIES also diverges.

   c $\sum_{n=0}^{\infty} \frac{(-2)^n-15n}{2^n-4}$

The dominating term on top and bottom is roughly 2^n. Again divide and compute the limit.

$$\lim_{n\to\infty} \frac{(-2)^n - 15n}{2^n - 4} = \frac{((-2)^n - 15n)\cdot\frac{1}{2^n}}{(2^n-4)\cdot\frac{1}{2^n}} = \frac{(-1)^n - \frac{15n}{2^n}}{1-\frac{4}{2^n}}$$

Using L'Hopital's Rule, $\lim_{n\to\infty}\frac{15n}{2^n} = \lim_{n\to\infty}\frac{15}{2^n \log 2} = 0$ (this is roughly how we know that an exponential term dominates a polynomial). Thus, as $n \to \infty$, this main limit bounces between -1 and 1, so it does not exist. Since the limit of the SEQUENCE does not exist, it cannot converge to zero, and thus, the SERIES must diverge.

d $\sum_{n=0}^{\infty}\frac{1}{n}$ and $\sum_{n=0}^{\infty}\frac{1}{n}$.
The limit of the each SEQUENCE goes to ZERO, but the first SERIES diverges, and the second SERIES converges (this can be seen from the p-test which we will discuss with the integral test). This shows that we can make no conclusion if the limit of the sequence is zero.

2. Geometric Series.

   A sequence of the form $\sum_{n=0}^{\infty} ar^n$ converges if $|r| < 1$ and diverges otherwise. If the series converges, then it converges to $\frac{a}{1-r}$. This also applies more generally

   Examples

   a $\sum_{n=3}^{\infty}\cdot\frac{1}{2^n} = \frac{1}{4}$.
   b $\sum_{n=1}^{\infty} 4\cdot\frac{1}{\pi^n} = \frac{4}{\pi-1}$.
   c $\sum_{n=1}^{\infty} 4\cdot\frac{e^{2n}}{\pi^n}$ diverges.

3. Telescoping series. These are series where the each term cancels out with part of the next. It is probably best to just do examples of these. This is the one case where we actually have to refer back to the definition of the derivative.

   a $\sum_{n=2}^{\infty}\frac{1}{n^2-1}$.

Note that from the techniques of chapter 8 (partial fractions), we have $\frac{1}{n^2-1} = \frac{1}{2(n-1)} - \frac{1}{2(n+1)}$. Thus,

$$\begin{aligned}
\sum_{n=2}^{\infty} \frac{1}{n^2-1} &= \sum_{n=2}^{\infty} \left(\frac{1}{2(n-1)} - \frac{1}{2(n+1)} \right) \\
&= \lim_{N\to\infty} \sum_{n=2}^{N} \frac{1}{2} \left(\frac{1}{(n-1)} - \frac{1}{(n+1)} \right) \\
&= \frac{1}{2} \lim_{N\to\infty} \left(\sum_{n=2}^{N} \frac{1}{(n-1)} - \sum_{n=2}^{N} \frac{1}{(n+1)} \right) \\
&= \frac{1}{2} \lim_{N\to\infty} \left(\sum_{n=0}^{N-2} \frac{1}{(n+1)} - \sum_{n=2}^{N} \frac{1}{(n+1)} \right) \\
&= \frac{1}{2} \lim_{N\to\infty} \left(1 + \frac{1}{2} + \sum_{n=2}^{N-2} \frac{1}{(n+1)} - \sum_{n=2}^{N-2} \frac{1}{(n+1)} - \frac{1}{N} - \frac{1}{N+1} \right) \\
&= \frac{1}{2} \lim_{N\to\infty} \left(1 + \frac{1}{2} - \frac{1}{N} - \frac{1}{N+1} \right) \\
&= \frac{1}{2} \left(1 + \frac{1}{2} \right) = \frac{3}{4}
\end{aligned}$$

b On your own show that $\sum_{n=0}^{\infty} \left(\frac{1}{\sqrt{3n+1}} - \frac{1}{\sqrt{3n+4}} \right) = 1$.

c Also, show that $\sum_{n=0}^{\infty} \left(\frac{1}{e^n} - \frac{1}{e^{n+2}} \right) = 1 - \frac{1}{e}$.

d As a challenge, show that

$$\sum_{n=0}^{\infty} \frac{\ln\left(\frac{n+2}{n+3}\right)}{\ln(n+2)\ln(n+3)} = \frac{1}{\ln(2)}$$

4. Integral test

In a very general sense, you can think of series and integrals being very similar. A result of this is the integral test. Let $f(x)$ be a continuous, decreasing, positive function on the interval $[c, \infty]$ such that $f(n) = a_n$. Then $\int_c^{\infty} f(x)dx$ has the same behavior (convergence/divergence) as $\sum_{n=c}^{\infty} a_n$.

Here are some examples:

a A special case of the integral test is the p-test. The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$, and diverges if $p \leq 1$. This follows directly from the integral test.

b We can show that $\sum_{n=2}^{\infty} \frac{1}{n \ln(n)}$ diverges by the integral test.

5. Direct Comparison Test (DCT)

   The DCT is basically just the Squeeze theorem for limits. Suppose that $0 \leq a_n \leq b_n$. The DCT says that IF $\sum_{n=c}^{\infty} a_n$ diverges, THEN $\sum_{n=c}^{\infty} b_n$ diverges, and IF $\sum_{n=c}^{\infty} b_n$ converges, THEN $\sum_{n=c}^{\infty} a_n$ converges.

   a We can show that $\sum_{n=2}^{\infty} \frac{\arctan(n)}{n^{\frac{3}{2}}}$ converges by comparing it to $\sum_{n=2}^{\infty} \frac{\frac{\pi}{2}}{n^{\frac{3}{2}}}$ which comverges by the p-test.

   b We can show that $\sum_{n=1}^{\infty} \frac{\sin(n)}{n^{\frac{3}{2}}}$ converges by comparing $\sum_{n=1}^{\infty} \frac{|\sin(n)|}{n^{\frac{3}{2}}}$ to $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$.

   c , we can show that $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n-1}}$ diverges by comparing it to $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$.

6. Limit Comparison Test (LCT)

   The LCT is a more accurate comparison test but requires a little bit more effort. Starting with a series $\sum_{n=c}^{\infty} a_n$, we try to find a similar series $\sum_{n=c}^{\infty} b_n$ where it is known whether or not it converges. The theorem states that

   a If $0 < \lim_{n\to\infty} \frac{a_n}{b_n} < \infty$, then the two series have the same behavior (both converge or both diverge.

   b If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$, then IF $\sum_{n=c}^{\infty} b_n$ diverges, THEN $\sum_{n=c}^{\infty} a_n$ diverges.

   c If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$, then IF $\sum_{n=c}^{\infty} b_n$ converges, THEN $\sum_{n=c}^{\infty} a_n$ converges.

   Be very careful about the one-way implications (note that they should make sense; so just try to blindly memorize them).

   a Use the LCT to show that $\sum_{n=1}^{\infty} \frac{\sqrt[3]{n^3+1}\sqrt{n^2+2}}{(n+3}^3$ diverges. Hint: Consider the dominating terms in each factor and compare to $\frac{1}{n}$.

7. Ratio Test

   The ratio test is checking to see if the series converges like a geometric series. Let $a_n > 0$, and let $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L$. If $L < 1$, then the series $\sum_{n=c}^{\infty} a_n$ comverges. If $L > 1$, then the series diverges. If $L = 1$, then the test is inconclusive.

a Show that $\sum_{n=0}^{\infty} \frac{n!}{n^n}$ converges. This one is a bit tricky. Try it and I'll do it during the review session.

8. Root Test

   The root test also checks to see if the series converges like a geometric series. Let $a_n > 0$, and let $\lim_{n\to\infty} \sqrt[n]{a_n} = L$. If $L < 1$, then the series $\sum_{n=c}^{\infty} a_n$ comverges. If $L > 1$, then the series diverges. If $L = 1$, then the test is inconclusive.

9. Alternating Series Test

   Most of the tests above require that our terms are positive. In this test we allow half of the terms to be negative.Let $\sum_{n=c}^{\infty} a_n = \sum_{n=c}^{\infty}(-1)^{n+1}u_n$. If the following are true,

   a $u_n \geq 0$,

   b $u_{n+1} \leq u_n$, and

   c $u_n \to 0$,

   then the series converges. This test ONLY shows CONVERGENCE. It will never imply divergence. However, note that if the third condition is not met, then we can apply the nth term test to show divergence. See the last quiz solution for an example.

   Also, recall, the notions absolutely convergent and conditionally convergent series. If $\sum_{n=c}^{\infty} a_n$ converges and $\sum_{n=c}^{\infty} |a_n|$ diverges, then the series is conditionally convergent. If both converge, we say $\sum_{n=c}^{\infty} a_n$ converges absolutely.

0.3 Taylor Series

I dont have time to write much on this topic, but I will close with some taylor series that you should know.

1. $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$.

2. $\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$.

3. $\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$.

4. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.