

## Exam 3 Review

# 1 Complex Numbers

We define  $i = \sqrt{-1}$ . Multiples of  $i$  are called imaginary numbers. Complex numbers are sums of real numbers and imaginary numbers ( $z = x + iy$ ). When  $x_1 + iy_1 = x_2 + iy_2$ , we actually have the two equalities:  $x_1 = x_2$  (the real part = the real part) and  $y_1 = y_2$  (the imaginary part = the imaginary part).

In chapter 11, we used Taylor Series to prove Euler's Identity:

$$e^{i\theta} = \cos \theta + i \sin \theta$$

This gives another way to represent polar coordinates:  $x + iy = re^{i\theta}$  where  $r = |x + iy| = \sqrt{x^2 + y^2}$ ,  $x = r \cos \theta$ , and  $y = r \sin \theta$ . This is exactly the same thing as polar coordinates. It is worth mentioning that since  $\cos$  and  $\sin$  are both  $2\pi$ -periodic (they repeat every  $2\pi$ ), we have that  $re^{i\theta} = re^{i(\theta+2k\pi)}$  where  $k$  is an integer.

In general, we want to use the polar representation to do computations, but the end result will be more useful in rectangular coordinates ( $x + iy$ ).

Euler's formula makes it much easier to multiply, divide, and take roots. Let  $z_1 = x_1 + iy_1 = r_1e^{i\theta_1}$  and  $z_2 = x_2 + iy_2 = r_2e^{i\theta_2}$ . It is much easier to compute the following:

$$\begin{aligned} z_1 z_2 &= r_1 e^{i\theta_1} r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1 + \theta_2)} \\ \frac{z_1}{z_2} &= \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)} \end{aligned}$$

than

$$\begin{aligned} z_1 z_2 &= (x_1 + iy_1)(x_2 + iy_2) = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1) \\ \frac{z_1}{z_2} &= \frac{x_1 + iy_1}{x_2 + iy_2} = \frac{x_1 + iy_1}{x_2 + iy_2} \cdot \frac{x_2 - iy_2}{x_2 - iy_2} \\ &= \frac{x_1 x_2 + y_1 y_2 + i(x_2 y_1 - x_1 y_2)}{x_2^2 - y_2^2}. \end{aligned}$$

Furthermore, without Euler's Formula, we would have no idea how to find  $\sqrt[n]{x + iy}$ . With Euler's Formula this is easy:

$$\sqrt[n]{re^{i\theta}} = \sqrt[n]{r} e^{i\frac{\theta}{n}}$$

The slightly confusing part is finding all of the roots. Since  $re^{i\theta} = re^{i(\theta+2k\pi)}$  for all integers  $k$ , we have that

$$\sqrt[n]{re^{i\theta}} = \sqrt[n]{re^{i(\theta+2k\pi)}} = \sqrt[n]{r}e^{i\frac{\theta+2k\pi}{n}}$$

We only have to compute these values for enough  $k$ 's. If we are taking the  $n$ th root, then we expect  $n$  values; hence, we may plug in  $k = 0, 1, \dots, n - 1$  to get all possible roots.

Another application of Euler's Formula is de Moivre's Formula:

$$(\cos \theta + i \sin \theta)^n = (e^{i\theta})^n = e^{in\theta} = \cos(n\theta) + i \sin(n\theta)$$

The double angle formulas can be derived from this by letting  $n = 2$ :

$$\begin{aligned} \cos(2\theta) + i \sin(2\theta) &= (\cos \theta + i \sin \theta)^2 \\ &= \cos^2 \theta - \sin^2 \theta + i(2 \sin \theta \cos \theta) \end{aligned}$$

Comparing real parts and imaginary parts, we get

$$\cos(2\theta) = \cos^2 \theta - \sin^2 \theta$$

$$\sin(2\theta) = 2 \sin \theta \cos \theta.$$

The last topic in this section dealt with the graphs of equations in the complex plane. The idea is to manipulate the equation until it becomes a rectangular equation that you recognize.

With all of these problems, see the book for examples. Try to do them all, and we'll talk about them on Tuesday.

## 2 First Order Differential Equations

These were relatively easy, but I'm sure many of you have forgotten the methods in the last 3-4 weeks. There are two types to be aware of.

The first are the separable equations of the form  $y' = f(x)g(y)$ . To solve, we separate  $y' = \frac{dy}{dx}$  to get  $\frac{dy}{g(y)} = f(x)dx$  and then integrate. These should be fairly routine, but definitely do some examples. Also, don't forget the  $+c$  if we want the general solution and know how to find an exact solution if you are given initial conditions.

The other type is of the form  $y' + P(x)y = Q(x)$ . To solve these, we first have to find the integrating factor  $v(x) = e^{\int P(x)dx}$ . Then we multiply  $v(x)$  to get  $v(x)y' + v(x)P(x)y = v(x)Q(x)$  and integrate. Note that  $\frac{d}{dx}(v(x)y) = v(x)y' + v(x)P(x)y$  (this always happens if  $v(x)$  is correct), so the equation becomes  $v(x)y = \int v(x)Q(x)dx$ .

Here is an example (p. 657, 10):  $x \frac{dy}{dx} = \frac{\cos x}{x} - 2y$  with  $x > 0$ . We must first get this equation into the right form:  $\frac{dy}{dx} + \frac{2}{x}y = \frac{\cos x}{x^2}$ . Thus  $P(x) = \frac{2}{x}$  and

$$v(x) = e^{\int P(x)dx} = e^{\int \frac{2}{x}dx} = e^{2 \ln x} = x^2.$$

Multiplying by  $v(x)$  we get  $x^2 \frac{dy}{dx} + 2xy = \cos x$ . Integrating both sides we get  $x^2 y = \sin(x)$  so  $y = \frac{\sin x}{x^2}$ .

Recall, that there were application problems in the homework. These probably wont be on the test but if you know how to do them, they should be easy. Read the section on Mixture Problems on pages 655-657 and try to do problems 25 and 26 again. We'll cover one of these in the review session if there is interest.

### **3 Second Order Differential Equations**

### **4 Conic Sections**

### **5 Polar Coordinates**