Math 222 Section 8.3 Problem 8

1. Introduction In this writeup, I would like to go over problem 8 from section 8.3. Before we get to the actual problem would like to state a general result that we will use to simplify the solution. I tried to state this result in one of the sections, and it was alluded to in class, but I think this might be what confused people the most. For completion, I will state the result and proof again.

   First, I will introduce some new notation. The symbol $|$ is used to mean 'divides'. Thus, we write $x|f(x)$ to mean that x divides $f(x)$. In this situation, we have that $f(x) = x \cdot f_1(x)$. For example, let $f(x) = x^4 + 3x^3 + 4x^2$. Then $x|f(x)$ (because $f(x) = x \cdot (x^3 + 3x^2 + 4x)$), $x^2|f(x)$, $(x+3)|f(x)$, and $(x+1)|f(x)$. We will also use the symbol $\nmid$ to mean 'does not divide'. Thus, we would write $x^3 \nmid f(x)$ to mean x^3 does not divide $f(x)$.

   Note that this notation works with regular integers (and that the definition of divisibility is basically the same). That is, if we write $7|56$, then this means 'seven divides fifty-six', and we know this because $56 = 7 \cdot 8$.

2. Lemma (this is just a math word that roughly means a small theorem that we will use later for our main result):

   Statement: Let $f(x) = \frac{g(x)}{h(x)}$ where $g(x)$ and $h(x)$ are polynomials (so $f(x)$ is a rational function). If $(x-a)^2|g(x)$, but $(x-a) \nmid h(x)$, then $(x-a)|f'(x)$.

   Proof: Since $(x-a)^2|g(x)$, this means that there exists $g_1(x)$ such that $g(x) = (x-a)^2 \cdot g_1(x)$. Hence, $f(x) = \frac{(x-a)^2 \cdot g_1(x)}{h(x)}$. Computing the derivative we have:

$$
\begin{aligned}
f'(x) &= \frac{d}{dx}\left(\frac{(x-a)^2 \cdot g_1(x)}{h(x)}\right) \\
&= \frac{\frac{d}{dx}((x-a)^2 \cdot g_1(x)) \cdot h(x) - (x-a)^2 \cdot g_1(x) \cdot h'(x)}{h(x)} \\
&= \frac{(2(x-a) \cdot g_1(x) + (x-a)^2 \cdot g_1'(x)) \cdot h(x) - (x-a)^2 \cdot g_1(x) \cdot h'(x)}{h(x)} \\
&= (x-a) \cdot \frac{(2 \cdot g_1(x) + (x-a) \cdot g_1'(x)) \cdot h(x) - (x-a) \cdot g_1(x) \cdot h'(x)}{h(x)}
\end{aligned}
$$

   The proof of the lemma is complete.

   Note that this lemma works for higher multiplicities of factors and higher derivatives. In other words, we can actually show (with roughly the same proof) that if $f(x) = \frac{g(x)}{h(x)}$ and $(x-a)^n|g(x)$, then let $f^{(n-1)}(x) = \frac{d^{n-1}}{dx^{n-1}}f(x)$. Then $(x-a)|f^{(n-1)}(x)$. Basically, we can think of this lemma as saying that each derivative takes away one factor of $(x-a)$ the denominator.

   Before we do problem 8, let's do an example of the lemma. Let $f(x) = \frac{g(x)}{h(x)} = \frac{x^4+3x^3+4x^2}{x^2-4}$. Since we have that $x^2|g(x)$ but $x^2 \nmid h(x)$, we should have if $f'(x) = \frac{g_1(x)}{h_1(x)}$, then $x|g_1(x)$, but $x \nmid h_1(x)$.

Computing the derivative, we get that

$$\begin{aligned} f'(x) &= \frac{(4x^3+9x^2+8x)(x^2-4)-(x^4+3x^3+4x^2)(2x)}{(x^2-4)^2} \\ &= \frac{(4x^5+9x^4+8x^3-16x^3-36x^2-32x)-(2x^5+6x^4+8x^3)}{(x^2-4)^2} \\ &= \frac{4x^5+9x^4+8x^3-16x^3-36x^2-32x-2x^5-6x^4-8x^3}{(x^2-4)^2} \\ &= \frac{2x^5+3x^4-16x^3-36x^2-32x}{(x^2-4)^2} = \frac{g_1(x)}{h_1(x)} \end{aligned}$$

Then, as expected, $x|g_1(x)$, but $x \nmid h_1(x)$.

3. Problem 8.

   Now that we are done with that huge digression, lets tackle the actual problem. We wanted to use partial fractions to rewrite $f(x) = \frac{t^4+9}{t^4+9t^2}$. I will break this problem down to simpler steps so it will be clearer.

   (a) First, since the degree of the numerator is greater than or equal to the degree of the denominator, we need to use long division to reduce the degree of the denominator. Hopefully, you all understand how to do this (I don't know a good way to type it). We have that

   $$\frac{t^4+9}{t^4+9t^2} = 1 + \frac{-9t^2+9}{t^4+9t^2} = 1 + \frac{-9t^2+9}{t^2(t^2+9)}.$$

   Now we want to find the partial fraction expansion of the second term. Since we have one linear factor of multiplicity 2 and one quadratic factor of multiplicity 1, we will have to use the Heaviside method and then a system of equations to find the variables in the following equation:

   $$\frac{-9t^2+9}{t^2(t^2+9)} = \frac{A}{t} + \frac{B}{t^2} + \frac{Ct+D}{t^2+9}.$$

   (b) The Heaviside Method.

   We will use this to find A and B. The first step is the same as the normal method. We multiply each side by t^2 so that the B term is left without a denominator. This gives

   $$\frac{-9t^2+9}{t^2+9} = At + B + \frac{(Ct+D)t^2}{t^2+9}.$$

   Now we take the limit as t goes to zero.

   $$\begin{aligned} \lim_{t\to 0} \frac{-9t^2+9}{t^2+9} &= \lim_{t\to 0} \left(At + B + \frac{(Ct+D)t^2}{t^2+9} \right) \\ 1 &= B. \end{aligned}$$

The next step in the Heaviside Method is to take the derivative of both sides. This is where it is very useful to apply the lemma from above. This tells us that

$$\frac{d}{dt}\left(\frac{(Ct+D)t^2}{t^2+9}\right) = \frac{t \cdot g(t)}{h(t)}$$

for some $g(t)$ and $h(t)$ where $t \nmid h(t)$. This means that

$$\lim_{t\to 0}\frac{d}{dt}\left(\frac{(Ct+D)t^2}{t^2+9}\right) = 0.$$

Thus we have that

$$\begin{aligned}
\frac{d}{dt}\left(\frac{-9t^2+9}{t^2+9}\right) &= \frac{d}{dt}\left(At+B+\frac{(Ct+D)t^2}{t^2+9}\right)\\
\frac{-18t(t^2+9)-2t(-9t^2+9)}{(t^2+9)^2} &= A+0+\frac{t \cdot g(t)}{h(t)}\\
\frac{-18t^3-172t+18t^3-18t)}{(t^2+9)^2} &= A+\frac{t \cdot g(t)}{h(t)}\\
\frac{180t)}{(t^2+9)^2} &= A+\frac{t \cdot g(t)}{h(t)}.
\end{aligned}$$

Now we again compute the limit:

$$\begin{aligned}
\lim_{t\to 0}\frac{180t)}{(t^2+9)^2} &= \lim_{t\to 0}\left(A+\frac{t \cdot g(t)}{h(t)}\right)\\
0 &= A
\end{aligned}$$

In other words the lemma prevented us from having to differentiate $\frac{(Ct+D)t^2}{t^2+9}$ because we knew that the limit would eliminate it anyway. Thats all it did for us here so it may seem that it was relatively useless. However, we should note that whenever we do problems like this this one (a linear factor of multiplicity greater than one together with atleast one other factor), then the lemma can be used again. In this light the lemma should seem useful.

Summing up our results from the Heaviside Method we have that

$$\frac{-9t^2+9}{t^2(t^2+9)} = \frac{1}{t^2}+\frac{Ct+D}{t^2+9}.$$

(c) System of equations.

We need to set up a system of equations to find C and D. We do this by multiplying by the denominator on the left. This will clear all of the denominators and we will be left with polynomials on each side of the equals sign.

$$\begin{aligned}
t^2(t^2+9)\cdot\frac{-9t^2+9}{t^2(t^2+9)} &= t^2(t^2+9)\cdot\left(\frac{1}{t^2}+\frac{Ct+D}{t^2+9}\right)\\
-9t^2+9 &= t^2+9+t^2(Ct+D)
\end{aligned}$$

$$(2)$$

$$-9t^2 + 9 = Ct^3 + (1 + D)t^2 + 9$$

(4)

This gives us that $C = 0$ and $D = -10$.
Thus the final answer is as follows:

$$\begin{aligned}\frac{t^4 + 9}{t^4 + 9t^2} &= 1 + \frac{A}{t} + \frac{B}{t^2} + \frac{Ct + D}{t^2 + 9} \\ &= 1 + \frac{1}{t^2} + \frac{-10}{t^2 + 9}\end{aligned}$$

4. Now, just for 'fun', assume we were given the problem

$$\int \frac{t^4 + 9}{t^4 + 9t^2} dt.$$

Using our previous work, we have that

$$\begin{aligned}\int \frac{t^4 + 9}{t^4 + 9t^2} dt &= \int \left(1 + \frac{1}{t^2} + \frac{-10}{t^2 + 9}\right) dt \\ &= \int dt + \int \frac{1}{t^2} dt + \int \frac{-10}{t^2 + 9} dt \\ &= t - \frac{1}{t} dt - 10 \int \frac{1}{9\left(\frac{t^2}{9} + 1\right)} dt \\ &= t - \frac{1}{t} dt - \frac{10}{9} \int \frac{1}{\frac{t^2}{9} + 1} dt\end{aligned}$$

To evaluate the last integral, we let $u = \frac{t}{3}$, so $du = \frac{1}{3} dt$ and $3du = dt$. Thus, the integral becomes

$$\begin{aligned}\int \frac{1}{\frac{t^2}{9} + 1} dt &= \int \frac{1}{u^2 + 1} \cdot 3 \cdot du \\ &= 3 \arctan(u) \\ &= 3 \arctan\left(\frac{t}{3}\right)\end{aligned}$$

Therefore,

$$\begin{aligned}\int \frac{t^4 + 9}{t^4 + 9t^2} dt &= t - \frac{1}{t} dt - \frac{10}{9} \int \frac{1}{\frac{t^2}{9} + 1} dt \\ &= t - \frac{1}{t} dt - \frac{10}{9} \cdot 3 \cdot \arctan\left(\frac{t}{3}\right) + c \\ &= t - \frac{1}{t} dt - \frac{10}{3} \arctan\left(\frac{t}{3}\right) + c\end{aligned}$$

Hopefully, this clears some stuff up. Let me know if you have questions or if you find mistakes.