Math 222 Quiz 3 Solution to Problem 1

The question that everyone struggled with asked you to determine whether $\int_1^\infty \frac{1}{e^x - 3^x} dx$ converges or diverges. Before doing the problems let's review the statements of the theorems that you might think to apply.

Direct Comparison Test (DCT) Let $f(x)$ and $g(x)$ be functions such that $0 \leq f(x) \leq g(x)$ for all x. Then the DCT states that $0 \leq \int_a^\infty f(x)dx \leq \int_a^\infty g(x)dx$. Specifically, (from the squeeze theorem) IF $\int_a^\infty f(x)dx$ diverges, THEN $\int_a^\infty g(x)dx$ diverges. Also, IF $\int_a^\infty g(x)dx$ converges, THEN $\int_a^\infty f(x)dx$ converges. Note that these are one-way implications.

Limit Comparison Test (LCT) Let $f(x)$ and $g(x)$ be functions such that $\lim_{x\to\infty} \frac{f(x)}{g(x)} = c$ for some finite positive constant c. Then the LCT states that $\int_a^\infty f(x)dx$ diverges IF AND ONLY IF $\int_a^\infty g(x)dx$ diverges. This is the same as saying that $\int_a^\infty g(x)dx$ converges IF AND ONLY IF $\int_a^\infty f(x)dx$ converges. Note that these are two-way implications.

Now we need to find a function to compare to $f(x) = \frac{1}{e^x - 3^x}$. Note that $e \approx 2.71828$, so $e < 3$, and $\frac{1}{e^x - 3^x} < 0$. This means we cannot use the DCT immediately (we need $f(x) \geq 0$). However, note that

$$\begin{aligned} \int_1^\infty \frac{1}{e^x - 3^x} dx &= -\int_1^\infty -\frac{1}{e^x - 3^x} dx \\ &= -\int_1^\infty \frac{1}{3^x - e^x} dx \end{aligned}$$

Because they are constant multiples of each other (the constant is -1), $\int_1^\infty \frac{1}{e^x - 3^x} dx$ converges if and only if $\int_1^\infty \frac{1}{3^x - e^x} dx$ does. As usual, lets try to throw away all but the dominating term and compare $\frac{1}{3^x - e^x}$ to $\frac{1}{3^x}$. We have that $0 \leq \frac{1}{3^x} < \frac{1}{3^x - e^x}$. Now we need to evaluate the integral of the new function. In extreme detail, it is as follows:

$$\begin{aligned} \int_1^\infty \frac{1}{3^x} dx &= \lim_{b\to\infty} \int_1^b \frac{1}{3^x} dx \\ &= \lim_{b\to\infty} \int_1^b 3^{-x} dx \\ &= \lim_{b\to\infty} \int_1^b (e^{\ln 3})^{-x} dx \\ &= \lim_{b\to\infty} \int_1^b e^{-x\ln 3} dx \\ &= \lim_{b\to\infty} \int_1^b e^{-x\ln 3} dx \\ &= \lim_{b\to\infty} -\ln 3 e^{-x\ln 3}|_1^b dx \\ &= \lim_{b\to\infty} -\frac{\ln 3}{e^{x\ln 3}}|_1^b dx \\ &= \lim_{b\to\infty} -\frac{\ln 3}{3^x}|_1^b dx \\ &= \lim_{b\to\infty} -\frac{\ln 3}{3} + \frac{\ln 3}{3^b} = -\frac{\ln 3}{3}. \end{aligned}$$

Thus the integral converges. This information does not help us with the DCT, so lets try to use the LCT.

We have two options here. We may compare the original function $\frac{1}{e^x-3^x}$ with $\frac{-1}{3^x}$ (two negative functions), or we may compare $\frac{1}{3^x-e^x}$ with $\frac{1}{3^x}$ (two positive functions). The end result will be the same because we already know that the respective integrals of $\frac{1}{e^x-3^x}$ and $\frac{1}{3^x-e^x}$ both either converge or diverge, and we just showed that $\int_1^\infty \frac{1}{3^x}dx$ converges (so $\int_1^\infty \frac{-1}{3^x}dx$ must as well). We will persue the first option. (Again, in great detail...)

$$\begin{aligned}
\lim_{x\to\infty} \frac{\frac{1}{e^x-3^x}}{\frac{-1}{3^x}} &= \lim_{x\to\infty} \frac{1}{e^x-3^x}\cdot\frac{3^x}{-1} \\
&= \lim_{x\to\infty} \frac{3^x}{3^x-e^x} \\
&= \lim_{x\to\infty} \frac{3^x}{3^x-e^x}\cdot\frac{\frac{1}{3^x}}{\frac{1}{3^x}} \\
&= \lim_{x\to\infty} \frac{1}{1-\frac{e^x}{3^x}} \\
&= \lim_{x\to\infty} \frac{1}{1-\left(\frac{e}{3}\right)^x} = 1
\end{aligned}$$

This limit is FINITE AND POSITIVE. Thus, by the LCT, the behavior of both integrals is the same, that is, $\int_1^\infty \frac{1}{e^x-3^x}dx$ converges.