Math 222 Quiz 5 Solutions

Name: ______________________________

SHOW WORK.

1. Determine whether the following series converges absolutely, converges conditionally, or diverges: $\sum_{n=1}^{\infty} \frac{(-1)^n n^4}{n^2\sqrt{n^5+4}}$. Explain your answer.

   Firs we can simplify the series to $\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{\sqrt{n^5+4}} = -\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^2}{\sqrt{n^5+4}}$. This is an alternating series in the form $\sum_{n=1}^{\infty}(-1)^{n+1}u_n$ where $u_n = \frac{n^2}{\sqrt{n^5+4}}$. To check for divergence using the Alternating Series Test, we need to check three things:

   (a) $u_n \geq 0$,
   (b) $u_{n+1} \leq u_n$, and
   (c) $\lim_{n\to\infty} u_n = 0$.

   We do this as follows.

   (a) Clearly, $\frac{n^2}{\sqrt{n^5+4}} \geq 0$.

   (b) Now we want to show that $\frac{(n+1)^2}{\sqrt{(n+1)^5+4}} \leq \frac{n^2}{\sqrt{n^5+4}}$.

   To see this we compute the derivative of $f(x) = \frac{x^2}{\sqrt{x^5+4}}$. We have that

$$f'(x) = \frac{2x(x^5+4) - x^2 \cdot 5x^4}{(x^5+4)^2} = \frac{-3x^6+8x}{(x^5+4)^2}$$

   Since the derivative is negative for $x \geq \sqrt[5]{\frac{8}{3}}$. Thus, after the second term, the sequence is decreasing, so (ignoring only finitely many terms), the condition is met. Recall, that this is okay, since we know that finitely many terms converge anyway.

   Another way to prove this part is to show that by simply manipulating the inequality. Each line in what follows is equivalent to the next line:

$$\begin{aligned}
\frac{(n+1)^2}{\sqrt{(n+1)^5+4}} &\leq \frac{n^2}{\sqrt{n^5+4}} \\
\frac{(n+1)^4}{(n+1)^5+4} &\leq \frac{n^4}{n^5+4} \\
(n+1)^4(n^5+4) &\leq n^4((n+1)^5+4) \\
n^5(n+1)^4 + 4(n+1)^4 &\leq n^4(n+1)^5 + 4n^4 \\
0 &\leq n^4(n+1)^4(n+1) - n^4(n+1)^4 n + 4n^4 - 4(n+1)^4 \\
0 &\leq n^4(n+1)^4((n+1)-n) + 4n^4 - 4(n^4+4n^3+6n^2+4n+1) \\
0 &\leq n^4(n+1)^4 - 4n^3 - 6n^2 - 4n - 1
\end{aligned}$$

   It should be clear that the last statement of true past a finite number of terms.

(c) Finally, we need to establish $\lim_{n\to\infty} u_n = 0$. This is done as follows:

$$\begin{aligned}
\lim_{n\to\infty} \frac{n^2}{\sqrt{n^5+4}} &= \lim_{n\to\infty} \frac{n^2\cdot\frac{1}{n^2}}{\sqrt{n^5+4}\cdot\frac{1}{n^2}} \\
&= \lim_{n\to\infty} \frac{1}{\sqrt{n^5+4\cdot\frac{1}{n^4}}} \\
&= \lim_{n\to\infty} \frac{1}{\sqrt{n+\cdot\frac{4}{n^4}}} = 0.
\end{aligned}$$

This shows that the series converges.

Now we need to check for absolute converges. We do this by taking the absolute value of the terms of the sequence, and trying to determine if the series $\sum_{n=1}^{\infty} \frac{n^4}{n^2\sqrt{n^5+4}} = \sum_{n=1}^{\infty} \frac{n^2}{\sqrt{n^5+4}}$ converges. We will use the Limit Comparison Test to show that this series diverges by comparing it to $\sum_{n=1}^{\infty}$

2. Recall that the Taylor series for $\sin(x)$ is $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$. Find the Taylor series expansion for $x\sin(-x^3)+x^4$. Determine where the series converges absolutely.

We have that:

$$\begin{aligned}
\sin(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \\
\sin(-x^3) &= \sum_{n=0}^{\infty} \frac{(-1)^n (-x^3)^{2n+1}}{(2n+1)!} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n (-1)^{2n+1} (x^3)^{2n+1}}{(2n+1)!} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n (-1) x^{6n+3}}{(2n+1)!} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{6n+3}}{(2n+1)!} \\
x\sin(-x^3) &= x\sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{6n+3}}{(2n+1)!} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{6n+4}}{(2n+1)!} \\
&= -x^4 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{6n+4}}{(2n+1)!} \\
x\sin(-x^3)+x^4 &= x^4 - x^4 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{6n+4}}{(2n+1)!}
\end{aligned}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{6n+4}}{(2n+1)!}$$

This is the Taylor series for $x\sin(-x^3)+x^4$. To determine convergence, we first check for absolute convergence using the ratio test.

$$\begin{aligned} \lim_{n\to\infty} \frac{\left|\frac{(-1)^{n+2}x^{6(n+1)+4}}{(2(n+1)+1)!}\right|}{\left|\frac{(-1)^{n+1}x^{6n+4}}{(2n+1)!}\right|} &= \lim_{n\to\infty} \frac{|x|^{6n+10}(2n+1)!}{|x|^{6n+4}(2n+3)!} \\ &= \lim_{n\to\infty} \frac{|x|^6}{(2n+3)(2n+2)} = 0 \end{aligned}$$

This limit is less than 1, for all values of x, so the series converges absolutely for all x.