

# ON THE $L$ -FUNCTIONS OF THE CURVES $y^2 = x^\ell + A$

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ABSTRACT. Let  $\ell > 3$  be an odd prime and  $A$  be an integer not divisible by  $\ell$ . Let  $C_A$  be the non-singular projective model over  $\mathbb{Q}$  of the affine curve  $C_A : y^2 = x^\ell + A$ . In this paper we use the theory of Hilbert modular Eisenstein series to obtain a formula for the central value of the Hasse-Weil  $L$ -function  $L(C_A, s)$  of the curve  $C_A$  when the cyclotomic field  $\mathbb{Q}(\zeta_\ell)$  has ideal class number 1.

## 1. INTRODUCTION AND STATEMENTS OF RESULTS

Let  $\ell > 3$  be an odd prime and  $A$  be an integer not divisible by  $\ell$ . Let  $C_A$  be the non-singular projective model over  $\mathbb{Q}$  of the affine curve

$$C_A : y^2 = x^\ell + A.$$

The arithmetic of the curves  $C_A$  and their Jacobians  $J_A$  has been studied extensively by M. Stoll and T. Yang in [S1, S2, S3, SY]. In [S3], Stoll investigated properties of the Hasse-Weil  $L$ -function  $L(C_A, s)$ . Let  $\zeta_\ell$  be a primitive  $\ell$ -th root of unity and let  $L = \mathbb{Q}(\zeta_\ell)$  and  $F = \mathbb{Q}(\zeta_\ell + \zeta_\ell^{-1})$ . The Jacobian  $J_A$  has complex multiplication by the ring of integers  $\mathcal{O}_L = \mathbb{Z}[\zeta_\ell]$  of  $L$ . A possible CM type for  $J_A$  is  $\{\sigma_1, \dots, \sigma_d\}$ , where  $\sigma_k : L \rightarrow \mathbb{C}$  is the embedding  $\zeta_\ell \mapsto e^{2\pi i k/\ell}$  and  $d = (\ell - 1)/2$  is the dimension of  $J_A$ . By [L, Section 1.7], this CM type is simple, which by [L, Theorem 1.3.5] implies that  $J_A$  is absolutely simple. By Milne [M, Theorem 4], one has

$$L(C_A, s) = L(J_A, s) = L(\eta_A, s)$$

for some Hecke Grössencharakter  $\eta_A$  of  $L$ . Here,  $L(C_A, s) = L(J_A, s)$  because both  $L$ -functions can be defined in terms of the Galois action on the first étale cohomology groups, and  $H_{\text{ét}}^1(C_A)$  and  $H_{\text{ét}}^1(J_A)$  are canonically isomorphic. Stoll explicitly determined the character  $\eta_A$  and calculated its root number.

In [Y1], Yang developed a powerful method for obtaining formulas for central values of Hecke  $L$ -functions using theta lifts. Stoll and Yang [SY] used the expression for  $\eta_A$  and methods similar to those in [Y1, RVY, Y2] to express the central value  $L(C_{A,5}, 1)$  of the curve  $C_{A,5} : y^2 = x^5 + A$  in terms of the value of a theta series for  $F = \mathbb{Q}(\sqrt{5})$  (depending on  $A$ ) evaluated at a CM point coming from  $L$ . Upon specializing to  $A = 1$ , they showed that  $L(C_{1,5}, 1) \neq 0$  and used a 2-descent as in [S2] to conclude that  $J_1(\mathbb{Q})$  is finite (of order 10).

In this paper, we will use the expression for  $\eta_A$  and the theory of Hilbert modular Eisenstein series to obtain a different type of formula for the central value  $L(C_A, 1)$  for all curves  $C_A : y^2 = x^\ell + A$  with  $\ell = 5, 7, 11, 13, 17$ , or 19 (this last condition insures that  $L = \mathbb{Q}(\zeta_\ell)$  has ideal class number 1). See Corollary 1.5. The formula for the central value should be

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1991 *Mathematics Subject Classification.* 11G40, 11G30, 11G15, 11F41.

*Key words and phrases.* Abelian variety, Central value, CM point, Hilbert modular Eisenstein series, Jacobian,  $L$ -function.

useful for numerical calculations. Moreover, we believe it is possible that the formula can be combined with an analysis of the zeros of Hilbert modular Eisenstein series similar to that in [Ba, H] to prove that  $L(C_A, 1) \neq 0$  for certain  $C_A$ .

We will derive the formula for the central value as follows. First, in Proposition 1.2 we relate  $L(C_A, s)$  to a twisted Hilbert modular Eisenstein series  $E(\phi, z, s)$  evaluated at a CM point. Along the way, we determine the automorphy factor arising from the action of  $GL_+(2, \mathcal{O}_F)$  on  $E(\phi, z, s)$ . Next, in Theorem 1.3 we compute the Fourier expansion of  $E(\phi, z, s)$  and use this in Corollary 1.4 to obtain an analytic expression for  $L(C_A, s)$ . Upon specializing to  $s = 1$  and computing certain integrals in a sum of divisors, we obtain the central value formula. Note that the analytic expression for  $L(C_A, s)$  provides an alternate proof that  $L(C_A, s)$  has an analytic continuation to an entire function on  $\mathbb{C}$ .

Although it is not our primary focus here, this paper also constitutes part of our effort to prove the Bloch-Beilinson conjecture for the curves  $C_A$ . Our strategy is to follow the original approach by Bloch [Bl] for elliptic curves with complex multiplication by the ring of integers of an imaginary quadratic field of class number 1. To do this, one must first relate the special value  $L(C_A, 2)$  to a Hilbert modular Kronecker-Eisenstein series. This follows from Proposition 1.2 (with  $s = 2$ ). Roughly speaking, one must then relate the Hilbert modular Kronecker-Eisenstein series to the image of a suitable ‘regulator’ on  $K_2(C_A)$ . We believe that the Fourier expansion in Theorem 1.3 will be useful for this purpose.

We now describe our main results. Throughout this paper we assume that  $L = \mathbb{Q}(\zeta_\ell)$  has ideal *class number 1*. Recall that if  $\ell > 3$  is an odd prime, the class number  $h(\mathbb{Q}(\zeta_\ell)) = 1$  if and only if  $\ell = 5, 7, 11, 13, 17$ , or  $19$  (see [W, Theorem 11.1]).

Let  $d := (\ell - 1)/2$ . Let  $\mathbb{H}$  be the complex upper half-plane, and let  $z = x + iy = (z_1, \dots, z_d) \in \mathbb{H}^d$ . Let  $F = \mathbb{Q}(\zeta_\ell + \zeta_\ell^{-1})$  be the maximal totally real subfield of  $L$ , let  $\{\tau_1, \dots, \tau_d\}$  be the  $d$  real embeddings of  $F$ , and define

$$N(az + b) = \prod_{j=1}^d (\tau_j(a)z_j + \tau_j(b)) \quad \text{for } a, b \in \mathcal{O}_F.$$

We then define the twisted Hilbert modular Eisenstein series

$$E(\phi, z, s) = \sum'_{(a,b) \in \mathcal{O}_F \times \mathcal{O}_F / \mathcal{O}_F^\times} \phi(a, b) N(az + b) |N(az + b)|^{-2s}, \quad z \in \mathbb{H}^d, \operatorname{Re}(s) > \frac{3}{2},$$

where  $'$  denotes the sum over nonzero elements,  $\mathcal{O}_F \times \mathcal{O}_F / \mathcal{O}_F^\times$  denotes the sum over a complete set of representatives  $(a, b) \in \mathcal{O}_F \times \mathcal{O}_F$  with respect to the equivalence relation  $(a, b) \sim (a', b')$  modulo  $\mathcal{O}_F^\times$  if and only if there exists a unit  $\epsilon \in \mathcal{O}_F^\times$  such that  $(a, b) = (\epsilon a', \epsilon b')$ , and  $\phi : \mathcal{O}_F \times \mathcal{O}_F \rightarrow \mathbb{C}$  is a doubly periodic function associated to the finite part  $\eta_{A,f} : (\mathcal{O}_L/\mathfrak{m})^\times \rightarrow \mathbb{C}^\times$  of the Grössencharakter  $\eta_A$  of  $L$  of modulus  $\mathfrak{m}$  determined by Stoll in [S3] (see Section 3).

Define the group

$$GL_+(2, \mathcal{O}_F) = \{A \in GL(2, \mathcal{O}_F) : \det(A) \text{ is totally positive}\}.$$

We will determine the automorphy factor arising from the action of  $GL_+(2, \mathcal{O}_F)$  on  $E(\phi, z, s)$ .

**Proposition 1.1.** *With notation as above we have*

$$E(\phi, Az, s) = \det(A)^{1-2s} N(\gamma z + \delta)^{-1} |N(\gamma z + \delta)|^{2s} E(\phi, z, s)$$

for all  $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in GL_+(2, \mathcal{O}_F)$ .

Fix the CM type  $\Phi = \{\sigma_1, \dots, \sigma_d\}$  for  $L/F$  and define the CM point

$$\mathbf{z}_{CM} := \Phi(\zeta_\ell) = (\sigma_1(\zeta_\ell), \dots, \sigma_d(\zeta_\ell)) = (\zeta_\ell, \dots, \zeta_\ell^d) \in \mathbb{H}^d.$$

Recall that the  $L$ -function of  $\eta_A$  is defined by

$$L(\eta_A, s) = \sum'_{\mathfrak{a} \subset \mathcal{O}_L} \eta_A(\mathfrak{a}) N_{L/\mathbb{Q}}(\mathfrak{a})^{-s}, \quad \text{Re}(s) > \frac{3}{2},$$

where the sum is over nonzero integral ideals. We will establish the following fundamental identity.

**Proposition 1.2.** *With notation as above we have*

$$L(\eta_A, s) = \frac{1}{|\mathcal{O}_L^\times : \mathcal{O}_F^\times|} E(\phi, \mathbf{z}_{CM}, s). \quad (1.1)$$

Let  $\{b_1, \dots, b_N\}$  be a complete set of coset representatives of  $\mathcal{O}_F/\mathfrak{m} \cap \mathcal{O}_F$ . Define the  $L$ -functions

$$L(\phi, s) = \sum'_{b \in \mathcal{O}_F/\mathcal{O}_F^\times} \phi(0, b) N(b) N_{F/\mathbb{Q}}((b))^{-s},$$

and

$$L(\phi_l, s) = \sum'_{a \in \mathcal{O}_F/\mathcal{O}_F^\times} \phi(a, b_l) N_{F/\mathbb{Q}}((a))^{-s}, \quad l = 1, \dots, N,$$

which are absolutely convergent for  $\text{Re}(s)$  sufficiently large.

Let  $\{a_1, \dots, a_N\}$  be (another) complete set of coset representatives of  $\mathcal{O}_F/\mathfrak{m} \cap \mathcal{O}_F$ , and let  $a(\mathfrak{m} \cap \mathcal{O}_F)$  denote the sum over a complete set of representatives  $a \in \mathcal{O}_F$  with respect to the equivalence relation  $a \sim a'$  modulo  $\mathfrak{m} \cap \mathcal{O}_F$  if and only if there exists a unit  $\epsilon \in \mathcal{O}_F^\times$  such that  $\epsilon \equiv 1 \pmod{\mathfrak{m} \cap \mathcal{O}_F}$  and  $a = \epsilon a'$ .

We will compute the Fourier expansion of  $E(\phi, z, s)$ .

**Theorem 1.3.** *The twisted Hilbert modular Eisenstein series  $E(\phi, z, s)$  has the Fourier expansion*

$$\begin{aligned} E(\phi, z, s) &= L(\phi, 2s) \\ &+ N_{F/\mathbb{Q}}(\mathfrak{m} \cap \mathcal{O}_F) D_F^{1/2} e^{\pi i d/2} N(y)^{2-2s} \left\{ \frac{2\pi \Gamma(2s-2) 2^{2-2s}}{\Gamma(s-1)\Gamma(s)} \right\}^d \sum_{l=1}^N L(\phi_l, 2s-2) \\ &+ N_{F/\mathbb{Q}}(\mathfrak{m} \cap \mathcal{O}_F) D_F^{1/2} e^{\pi i d/2} N(y)^{2-2s} \\ &\times \sum_{k=1}^N \sum_{l=1}^N \phi(a_k, b_l) \sum'_{\tilde{g} \in (\mathfrak{m} \cap \mathcal{O}_F)^*} \sigma_{\tilde{g}}(k, l, y, s) \exp\{2\pi i T(\tilde{g}x)\} \end{aligned}$$

where the sum of divisors  $\sigma_{\tilde{g}}(k, l, y, s)$  is defined by

$$\sigma_{\tilde{g}}(k, l, y, s) = \sum_{\substack{\tilde{g}=ag \\ g \in (\mathfrak{m} \cap \mathcal{O}_F)^* \\ a \equiv a_k \pmod{\mathfrak{m} \cap \mathcal{O}_F} \\ a \in (\mathfrak{m} \cap \mathcal{O}_F)}} \prod_{j=1}^d h(2\pi\tau_j(agy); s-1, s) N(a)^{2-2s} \exp\{2\pi iT(gb_l)\}.$$

Here  $D_F$  is the absolute discriminant of  $F$ ,  $T$  is the trace,  $(\mathfrak{m} \cap \mathcal{O}_F)^*$  is the dual of the lattice  $\mathfrak{m} \cap \mathcal{O}_F$ , and

$$h(y; s-1, s) := \int_{\mathbb{R}} (1-it) |1-it|^{-2s} e^{-ity} dt, \quad \operatorname{Re}(s) > 2, \quad y \in \mathbb{R},$$

which extends to  $\mathbb{C}$  by meromorphic continuation (it is entire if  $y \neq 0$ ).

By combining Theorem 1.3 with the fundamental identity (1.1) and the fact that the  $L$ -series  $L(\phi, s)$  and  $L(\phi_l, s)$  have analytic continuations to entire functions on  $\mathbb{C}$  (see [F, pg. 171]), we obtain the following corollary.

**Corollary 1.4.** *The  $L$ -function  $L(\eta_A, s)$  can be expressed as*

$$\begin{aligned} |\mathcal{O}_L^\times : \mathcal{O}_F^\times| L(\eta_A, s) &= L(\phi, 2s) \\ &+ N_{F/\mathbb{Q}}(\mathfrak{m} \cap \mathcal{O}_F) D_F^{1/2} e^{\pi id/2} N(\mathbf{y}_{CM})^{2-2s} \left\{ \frac{2\pi\Gamma(2s-2)2^{2-2s}}{\Gamma(s-1)\Gamma(s)} \right\}^d \sum_{l=1}^N L(\phi_l, 2s-2) \\ &+ N_{F/\mathbb{Q}}(\mathfrak{m} \cap \mathcal{O}_F) D_F^{1/2} e^{\pi id/2} N(\mathbf{y}_{CM})^{2-2s} \\ &\times \sum_{k=1}^N \sum_{l=1}^N \phi(a_k, b_l) \sum'_{\tilde{g} \in (\mathfrak{m} \cap \mathcal{O}_F)^*} \sigma_{\tilde{g}}(k, l, \mathbf{y}_{CM}, s) \exp\{2\pi iT(\tilde{g}\mathbf{x}_{CM})\}. \end{aligned}$$

Furthermore, this expression provides an analytic continuation of  $L(\eta_A, s)$  to an entire function on  $\mathbb{C}$ .

By computing the integrals  $h(2\pi\tau_j(a\mathbf{g}\mathbf{y}_{CM}); 0, 1)$  appearing in the sum of divisors  $\sigma_{\tilde{g}}(k, l, \mathbf{y}_{CM}, 1)$  we will obtain the following formula for the central value  $L(C_A, 1)$ .

**Corollary 1.5.** *The central value of the  $L$ -function  $L(C_A, s)$  is given by*

$$\begin{aligned} |\mathcal{O}_L^\times : \mathcal{O}_F^\times| L(C_A, 1) &= L(\phi, 2) + N_{F/\mathbb{Q}}(\mathfrak{m} \cap \mathcal{O}_F) D_F^{1/2} e^{\pi id/2} (2\pi)^d \sum_{l=1}^N L(\phi_l, 0) \\ &+ N_{F/\mathbb{Q}}(\mathfrak{m} \cap \mathcal{O}_F) D_F^{1/2} e^{\pi id/2} \\ &\times \sum_{k,l=1}^N \phi(a_k, b_l) \sum'_{\tilde{g} \in (\mathfrak{m} \cap \mathcal{O}_F)^*} \sigma_{\tilde{g}}(k, l, \mathbf{y}_{CM}, 1) \exp\{2\pi iT(\tilde{g}\mathbf{x}_{CM})\} \end{aligned}$$

where

$$\sigma_{\tilde{g}}(k, l, \mathbf{y}_{CM}, 1) = \sum_{\substack{\tilde{g}=ag \\ g \in (\mathfrak{m} \cap \mathcal{O}_F)^* \\ a \equiv a_k \pmod{\mathfrak{m} \cap \mathcal{O}_F} \\ a \in (\mathfrak{m} \cap \mathcal{O}_F)}} \chi(a\mathbf{g}\mathbf{y}_{CM}) \exp\{2\pi iT(gb_l)\}$$

and

$$\chi(\mathbf{ag}\mathbf{y}_{CM}) = \begin{cases} 0, & \text{if } \tau_j(\mathbf{ag}) > 0 \text{ for some } j, \\ (2\pi)^d \exp\{2\pi T(\mathbf{ag}\mathbf{y}_{CM})\}, & \text{if } \tau_j(\mathbf{ag}) < 0 \text{ for } j = 1, \dots, d. \end{cases}$$

Finally, we will make Corollary 1.5 more explicit for the curve  $C_{1,5} : y^2 = x^5 + 1$ .

**Corollary 1.6.** *The central value of the  $L$ -function  $L(C_{1,5}, s)$  is given by*

$$\begin{aligned} |\mathcal{O}_L^\times : \mathcal{O}_F^\times| L(C_{1,5}, 1) &= L(\phi, 2) + 20\sqrt{5}e^{\pi i} (2\pi)^2 \sum_{l=1}^N L(\phi_l, 0) \\ &+ 20\sqrt{5}e^{\pi i} \sum_{k=1}^N \sum_{l=1}^N \phi(a_k, b_l) \sum'_{\tilde{g} \in (2\sqrt{5}\mathcal{O}_F)^*} \sigma_{\tilde{g}}(k, l, \mathbf{y}_{CM}, 1) \exp\{2\pi i T(\tilde{g}\mathbf{x}_{CM})\} \end{aligned}$$

where

$$\sigma_{\tilde{g}}(k, l, \mathbf{y}_{CM}, 1) = \sum_{\substack{\tilde{g}=\mathbf{ag} \\ g \in (2\sqrt{5}\mathcal{O}_F)^* \\ a \equiv a_k \pmod{2\sqrt{5}\mathcal{O}_F} \\ a \in 2\sqrt{5}\mathcal{O}_F}} \chi(\mathbf{ag}\mathbf{y}_{CM}) \exp\{2\pi i T(gb_l)\}.$$

As discussed, Stoll and Yang [SY] used different methods to prove that  $L(C_{1,5}, 1) > 0$ .

**Acknowledgments.** I would like to thank Fernando Rodriguez-Villegas for several helpful discussions, and the referee and editorial advisor for detailed comments leading to improvements in the paper. The author was supported by the IHÉS and CRM-Montréal during the preparation of this work.

## 2. THE $L$ -FUNCTIONS OF THE CURVES $y^2 = x^\ell + A$

As discussed in the introduction, by Milne [M, Theorem 4] the  $L$ -function of  $C_A$  is a Hecke  $L$ -function, and Stoll [S3] explicitly determined the Grössencharakter  $\eta_A$  appearing in the Hecke  $L$ -function. The Grössencharakter  $\eta_A$  is a multiplicative character on the group of ideals prime to its modulus  $\mathfrak{m}$ . We extend  $\eta_A$  by zero to all nonzero integral ideals of  $L$ . Let  $'$  denote the sum over nonzero elements. The Hecke  $L$ -series of  $\eta_A$  is defined by

$$L(\eta_A, s) = \sum'_{\mathfrak{A} \subset \mathcal{O}_L} \eta_A(\mathfrak{A}) N_{L/\mathbb{Q}}(\mathfrak{A})^{-s}, \quad \operatorname{Re}(s) > 3/2, \quad (2.1)$$

where the sum is over (nonzero) integral ideals  $\mathfrak{A}$  of  $L$  and  $N_{L/\mathbb{Q}}(\mathfrak{A}) = |\mathcal{O}_L : \mathfrak{A}|$ . If  $(\alpha) \subset \mathcal{O}_L$  is principal then  $N_{L/\mathbb{Q}}((\alpha)) = |\prod_{j=1}^{\ell-1} \sigma_j(\alpha)|$ , where the product is over the automorphisms  $\sigma_j \in \operatorname{Gal}(L/\mathbb{Q})$ .

Let  $\alpha, \beta \in \mathcal{O}_L$ . We say that  $\alpha$  and  $\beta$  are associate modulo  $\mathcal{O}_L^\times$  if there exists a unit  $\epsilon \in \mathcal{O}_L^\times$  such that  $\alpha = \epsilon\beta$ . It is clear that the principal ideals  $(\alpha) = (\beta)$  if and only if  $\alpha$  and  $\beta$  are associate modulo  $\mathcal{O}_L^\times$ .

Since we have assumed that  $L$  has ideal class number 1,  $\mathcal{O}_L$  is a principal ideal domain. It follows that there is a one-to-one correspondence between ideals of  $\mathcal{O}_L$  and principal ideals generated by elements of  $\mathcal{O}_L$  which are non-associate modulo  $\mathcal{O}_L^\times$ . Let  $\mathcal{O}_L/\mathcal{O}_L^\times$  denote a complete set of elements of  $\mathcal{O}_L$  which are non-associate modulo  $\mathcal{O}_L^\times$ . Then using that

$$\eta_A((\epsilon\alpha)) = \eta_A((\alpha)) \quad \text{for } \epsilon \in \mathcal{O}_L^\times,$$

and

$$N_{L/\mathbb{Q}}((\epsilon\alpha)) = N_{L/\mathbb{Q}}((\alpha)),$$

we can express the  $L$ -function (2.1) as

$$\begin{aligned} L(\eta_A, s) &= \sum'_{\alpha \in \mathcal{O}_L/\mathcal{O}_L^\times} \eta_A((\alpha)) N_{L/\mathbb{Q}}((\alpha))^{-s} \\ &= \frac{1}{|\mathcal{O}_L^\times : \mathcal{O}_F^\times|} \sum'_{\alpha \in \mathcal{O}_L/\mathcal{O}_F^\times} \eta_A((\alpha)) N_{L/\mathbb{Q}}((\alpha))^{-s}. \end{aligned} \quad (2.2)$$

### 3. A DOUBLY PERIODIC FUNCTION ASSOCIATED TO $\eta_A$

Let  $\mathfrak{m}$  be an integral ideal of  $L$ , and let  $J^\mathfrak{m}$  be the group of all ideals of  $L$  which are relatively prime to  $\mathfrak{m}$ . Define

$$\mathcal{O}_L^{(\mathfrak{m})} = \{\alpha \in \mathcal{O}_L : (\alpha, \mathfrak{m}) = 1\}.$$

Since  $\eta_A$  is a Grössencharakter modulo  $\mathfrak{m}$ , we know that  $\eta_A$  is a multiplicative character

$$\eta_A : J^\mathfrak{m} \rightarrow \mathbb{C}^\times$$

for which there exists a pair of characters

$$\eta_{A,f} : (\mathcal{O}_L/\mathfrak{m})^\times \rightarrow \mathbb{C}^\times, \quad \eta_\infty : L_\infty^\times \rightarrow \mathbb{C}^\times$$

such that

$$\eta_A((\alpha)) = \eta_{A,f}(\alpha) \eta_\infty(\alpha)$$

for all  $\alpha \in \mathcal{O}_L^{(\mathfrak{m})}$ . The ideal  $\mathfrak{m}$  is called the *modulus* of  $\eta_A$ .

Write  $A = 2^e B$  with  $B$  odd. By [S3, Proposition 3.3] the modulus of  $\eta_A$  is

$$\mathfrak{m} = 2^3(1 - \zeta_\ell)^2 B.$$

In [S3, Section 3], Stoll explicitly determined the characters  $\eta_{A,f}$  and  $\eta_\infty$ .

It is known that  $\mathcal{O}_L = \mathbb{Z}[\zeta_\ell]$  is a free, rank 2 module over  $\mathcal{O}_F = \mathbb{Z}[\zeta_\ell + \zeta_\ell^{-1}]$  with basis  $\{1, \zeta_\ell\}$  (see [W, pg. 44]). Write

$$\mathcal{O}_L = \mathcal{O}_F \zeta_\ell + \mathcal{O}_F.$$

**Proposition 3.1.** *Define the function*

$$\phi : \mathcal{O}_F \times \mathcal{O}_F \rightarrow \mathbb{C}$$

by

$$\phi(a, b) = \begin{cases} \eta_{A,f}(a\zeta_\ell + b), & (a\zeta_\ell + b, \mathfrak{m}) = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Then  $\phi$  is periodic modulo  $\mathfrak{m} \cap \mathcal{O}_F$  in each argument.

*Proof.* Observe that  $a \equiv a' \pmod{\mathfrak{m} \cap \mathcal{O}_F}$  implies  $a\zeta_\ell + b \equiv a'\zeta_\ell + b \pmod{(\mathfrak{m} \cap \mathcal{O}_F)\mathcal{O}_L}$  implies  $a\zeta_\ell + b \equiv a'\zeta_\ell + b \pmod{\mathfrak{m}}$ . In addition, if  $a\zeta_\ell + b \equiv a'\zeta_\ell + b \pmod{\mathfrak{m}}$  and  $(a\zeta_\ell + b, \mathfrak{m}) = 1$ , then  $(a'\zeta_\ell + b, \mathfrak{m}) = 1$ . Because each element  $\alpha \in \mathcal{O}_L^{(\mathfrak{m})}$  determines a well-defined class in  $(\mathcal{O}_L/\mathfrak{m})^\times$ , and  $\eta_{A,f} : (\mathcal{O}_L/\mathfrak{m})^\times \rightarrow \mathbb{C}^\times$ , we conclude that  $\phi(a, b) = \eta_{A,f}(a\zeta_\ell + b) = \eta_{A,f}(a'\zeta_\ell + b) = \phi(a', b)$ . On the other hand, if  $(a\zeta_\ell + b, \mathfrak{m}) \neq 1$ , we have  $\phi(a, b) = \phi(a', b) = 0$ . A similar argument works for  $\phi(a, \cdot)$ .  $\square$

#### 4. PROOF OF PROPOSITION 1.1

Recall that the twisted Hilbert modular Eisenstein series is defined by

$$E(\phi, z, s) = \sum'_{(a,b) \in \mathcal{O}_F \times \mathcal{O}_F / \mathcal{O}_F^\times} \phi(a, b) N(az + b) |N(az + b)|^{-2s}, \quad z \in \mathbb{H}^d, \quad \operatorname{Re}(s) > \frac{3}{2}.$$

It can be shown using the definition of the character  $\eta_{A,f}$  in [S3, Section 3] that  $\phi(\epsilon a, \epsilon b) = N(\epsilon)\phi(a, b)$  for all  $\epsilon \in \mathcal{O}_F^\times$ , and hence the Eisenstein series  $E(\phi, z, s)$  is well-defined.

We want to compute the automorphy factor arising from the action of

$$GL_+(2, \mathcal{O}_F) = \{A \in GL(2, \mathcal{O}_F) : \det(A) \text{ is totally positive}\}$$

on  $E(\phi, z, s)$ .

Let  $S = \{\alpha_1, \dots, \alpha_M\}$  be a complete set of coset representatives of  $(\mathcal{O}_L / (\mathfrak{m} \cap \mathcal{O}_F)\mathcal{O}_L)^\times$ .

**Lemma 4.1.** *Let  $\alpha \in \mathcal{O}_L$ ,  $\alpha_i \in S$ , and fix decompositions  $\alpha = a\zeta_\ell + b$ ,  $\alpha_i = a_i\zeta_\ell + b_i$ , where  $a, b, a_i, b_i \in \mathcal{O}_F$ . Then  $\alpha \equiv \alpha_i \pmod{(\mathfrak{m} \cap \mathcal{O}_F)\mathcal{O}_L}$  if and only if  $a \equiv a_i \pmod{\mathfrak{m} \cap \mathcal{O}_F}$  and  $b \equiv b_i \pmod{\mathfrak{m} \cap \mathcal{O}_F}$ .*

*Proof.*  $\Rightarrow$  Using the decomposition

$$\mathcal{O}_L = \mathcal{O}_F\zeta_\ell + \mathcal{O}_F$$

we obtain

$$(\mathfrak{m} \cap \mathcal{O}_F)\mathcal{O}_L = (\mathfrak{m} \cap \mathcal{O}_F)\mathcal{O}_F\zeta_\ell + (\mathfrak{m} \cap \mathcal{O}_F)\mathcal{O}_F.$$

Then  $\alpha \equiv \alpha_i \pmod{(\mathfrak{m} \cap \mathcal{O}_F)\mathcal{O}_L}$  if and only if  $\alpha - \alpha_i \in (\mathfrak{m} \cap \mathcal{O}_F)\mathcal{O}_L$  which implies  $\alpha - \alpha_i = \tilde{a}\zeta_\ell + \tilde{b}$  for some  $\tilde{a}, \tilde{b} \in \mathfrak{m} \cap \mathcal{O}_F$ . From the decompositions for  $\alpha, \alpha_i$  in the statement of the lemma, we also have  $\alpha - \alpha_i = (a - a_i)\zeta_\ell + (b - b_i)$  where  $a - a_i, b - b_i \in \mathcal{O}_F$ . Since  $\{1, \zeta_\ell\}$  is a basis for the module  $\mathcal{O}_L$  over  $\mathcal{O}_F$ , it follows that  $\tilde{a} = a - a_i$  and  $\tilde{b} = b - b_i$  which implies  $a \equiv a_i \pmod{\mathfrak{m} \cap \mathcal{O}_F}$  and  $b \equiv b_i \pmod{\mathfrak{m} \cap \mathcal{O}_F}$ .

$\Leftarrow$  By hypothesis,  $a - a_i, b - b_i \in \mathfrak{m} \cap \mathcal{O}_F \subset (\mathfrak{m} \cap \mathcal{O}_F)\mathcal{O}_L$ . Since  $(\mathfrak{m} \cap \mathcal{O}_F)\mathcal{O}_L$  is an ideal of  $\mathcal{O}_L$ , it follows that  $\alpha - \alpha_i = (a - a_i)\zeta_\ell + (b - b_i) \in (\mathfrak{m} \cap \mathcal{O}_F)\mathcal{O}_L$  which implies  $\alpha \equiv \alpha_i \pmod{(\mathfrak{m} \cap \mathcal{O}_F)\mathcal{O}_L}$ .  $\square$

**Lemma 4.2.** *Suppose that  $\alpha, \gamma \in \mathcal{O}_L^{((\mathfrak{m} \cap \mathcal{O}_F)\mathcal{O}_L)}$ ,  $\alpha \equiv \gamma \pmod{(\mathfrak{m} \cap \mathcal{O}_F)\mathcal{O}_L}$ , and  $\epsilon \in \mathcal{O}_F^\times$  is such that  $\alpha = \epsilon\gamma$ . Then  $\epsilon \equiv 1 \pmod{\mathfrak{m} \cap \mathcal{O}_F}$ .*

*Proof.* Observe that  $\alpha \equiv \gamma \pmod{(\mathfrak{m} \cap \mathcal{O}_F)\mathcal{O}_L}$  implies  $\epsilon\gamma \equiv \gamma \pmod{(\mathfrak{m} \cap \mathcal{O}_F)\mathcal{O}_L}$  implies  $(\epsilon - 1)\gamma \in (\mathfrak{m} \cap \mathcal{O}_F)\mathcal{O}_L$  implies  $(\mathfrak{m} \cap \mathcal{O}_F)\mathcal{O}_L | (\epsilon - 1)\gamma$ . However,  $(\gamma, (\mathfrak{m} \cap \mathcal{O}_F)\mathcal{O}_L) = 1$ , so  $\epsilon - 1 \in (\mathfrak{m} \cap \mathcal{O}_F)\mathcal{O}_L$ . Since  $\epsilon - 1 \in \mathcal{O}_F$ , we have  $\epsilon - 1 \in (\mathfrak{m} \cap \mathcal{O}_F)\mathcal{O}_L \cap \mathcal{O}_F = \mathfrak{m} \cap \mathcal{O}_F$ .  $\square$

Let  $(a, b)(\mathfrak{m} \cap \mathcal{O}_F)$  denote the sum over a complete set of representatives  $(a, b) \in \mathcal{O}_F \times \mathcal{O}_F$  with respect to the equivalence relation  $(a, b) \sim (a', b')$  modulo  $\mathfrak{m} \cap \mathcal{O}_F$  if and only if there exists a unit  $\epsilon \in \mathcal{O}_F^\times$  such that  $\epsilon \equiv 1 \pmod{\mathfrak{m} \cap \mathcal{O}_F}$  and  $a = \epsilon a', b = \epsilon b'$ . Define the Hilbert modular Eisenstein series of level  $\mathfrak{m} \cap \mathcal{O}_F$  by

$$E_i(z, s) = \sum'_{\substack{(a,b) \in \mathcal{O}_F \times \mathcal{O}_F \\ a \equiv a_i \pmod{\mathfrak{m} \cap \mathcal{O}_F} \\ b \equiv b_i \pmod{\mathfrak{m} \cap \mathcal{O}_F} \\ (a,b)(\mathfrak{m} \cap \mathcal{O}_F)}} N(az + b) |N(az + b)|^{-2s}, \quad z \in \mathbb{H}^d, \quad \operatorname{Re}(s) > \frac{3}{2}.$$

See [F, pg. 161].

Now, using the periodicity of  $\phi$  and Lemmas 4.1 and 4.2 we obtain the decomposition

$$E(\phi, z, s) = \sum_{i=1}^M \phi(a_i, b_i) E_i(z, s). \quad (4.1)$$

The automorphy factor in the proposition can now be obtained by combining (4.1) with [F, Lemma 4.2].

### 5. AN ALGEBRAIC IDENTITY FOR $\eta_\infty N_{L/\mathbb{Q}}$

Let  $\text{Gal}(L/\mathbb{Q}) = \{\sigma_1, \dots, \sigma_{\ell-1}\}$  where  $\sigma_j(\zeta_\ell) = \zeta_\ell^j$  for  $j = 1, \dots, \ell - 1$ . We have

$$\overline{\sigma_j} = \sigma_j^{-1} = \sigma_{\ell-j} \quad \text{for } j = 1, \dots, d = \frac{\ell-1}{2}.$$

By considering the action of the automorphisms  $\{\sigma_j\}_{j=1}^{\ell-1}$  on the generator  $\zeta_\ell + \zeta_\ell^{-1}$  of  $F/\mathbb{Q}$ , we find that  $\tau_j := \sigma_j|_F = \sigma_{\ell-j}|_F$  for  $j = 1, \dots, d$ .

Let  $\mathbb{H}$  be the complex upper half-plane, let  $z = x + iy = (z_1, \dots, z_d) \in \mathbb{H}^d$ , let  $\{\tau_1, \dots, \tau_d\}$  be the  $d$  real embeddings of  $F$ , and define

$$N(az + b) = \prod_{j=1}^d (\tau_j(a)z_j + \tau_j(b)) \quad \text{for } a, b \in \mathcal{O}_F.$$

Fix the CM type  $\Phi = \{\sigma_1, \dots, \sigma_d\}$  for  $L/F$  and define the CM point

$$\mathbf{z}_{CM} := \Phi(\zeta_\ell) = (\sigma_1(\zeta_\ell), \dots, \sigma_d(\zeta_\ell)) = (\zeta_\ell, \dots, \zeta_\ell^d) \in \mathbb{H}^d.$$

By [S3, Section 3] the character  $\eta_\infty$  is given by

$$\eta_\infty(\alpha) = \alpha^\theta$$

where

$$\theta = \sum_{j=1}^d \sigma_j \in \mathbb{Z}[\text{Gal}(L/\mathbb{Q})]$$

is the Sticklerberger element associated to the CM type  $\Phi$ .

**Lemma 5.1.** *With notation as above we have*

$$\eta_\infty(a\zeta_\ell + b) N_{L/\mathbb{Q}}(a\zeta_\ell + b) = N(\mathbf{a}\mathbf{z}_{CM} + b) |N(\mathbf{a}\mathbf{z}_{CM} + b)|^{-2s}.$$

*Proof.* First we compute

$$\begin{aligned} \eta_\infty(a\zeta_\ell + b) &= (a\zeta_\ell + b)^\theta \\ &= \prod_{j=1}^d \sigma_j(a\zeta_\ell + b) \\ &= \prod_{j=1}^d (\tau_j(a)\sigma_j(\zeta_\ell) + \tau_j(a)) \\ &= N(\mathbf{a}\mathbf{z}_{CM} + b). \end{aligned}$$

Next, using  $\overline{\sigma_j} = \sigma_{\ell-j}$ ,  $\tau_j := \sigma_j|_F = \sigma_{\ell-j}|_F$ , and that  $F/\mathbb{Q}$  is totally real, we compute

$$\begin{aligned}
N_{L/\mathbb{Q}}((a\zeta_\ell + b)) &= \prod_{j=1}^{\ell-1} \sigma_j(a\zeta_\ell + b) \\
&= \prod_{j=1}^d (\sigma_j(a)\sigma_j(\zeta_\ell) + \sigma_j(b))(\sigma_{\ell-j}(a)\sigma_{\ell-j}(\zeta_\ell) + \sigma_{\ell-j}(b)) \\
&= \prod_{j=1}^d (\tau_j(a)\sigma_j(\zeta_\ell) + \tau_j(b))(\tau_j(a)\overline{\sigma_j}(\zeta_\ell) + \tau_j(b)) \\
&= \prod_{j=1}^d (\tau_j(a)\sigma_j(\zeta_\ell) + \tau_j(b))\overline{(\tau_j(a)\sigma_j(\zeta_\ell) + \tau_j(b))} \\
&= \prod_{j=1}^d |(\tau_j(a)\sigma_j(\zeta_\ell) + \tau_j(b))|^2 \\
&= \left| \prod_{j=1}^d (\tau_j(a)\sigma_j(\zeta_\ell) + \tau_j(b)) \right|^2 \\
&= |N(a\mathbf{z}_{CM} + b)|^2.
\end{aligned}$$

□

## 6. PROOF OF PROPOSITION 1.2

We are now in a position to express the  $L$ -function  $L(\eta_A, s)$  in terms of the twisted Hilbert modular Eisenstein series  $E(\phi, z, s)$  evaluated at the CM point  $\mathbf{z}_{CM} = (\zeta_\ell, \dots, \zeta_\ell^d) \in \mathbb{H}^d$ .

Using (2.2), the decompositions  $\eta_A = \eta_{A,f}\eta_\infty$  and  $\mathcal{O}_L = \mathcal{O}_F\zeta_\ell + \mathcal{O}_F$ , and Lemma 5.1, we obtain

$$\begin{aligned}
L(\eta_A, s) &= \frac{1}{|\mathcal{O}_L^\times : \mathcal{O}_F^\times|} \sum'_{\alpha \in \mathcal{O}_L/\mathcal{O}_F^\times} \eta_A((\alpha)) N_{L/\mathbb{Q}}((\alpha))^{-s} \\
&= \frac{1}{|\mathcal{O}_L^\times : \mathcal{O}_F^\times|} \sum'_{(a,b) \in \mathcal{O}_F \times \mathcal{O}_F/\mathcal{O}_F^\times} \eta_{A,f}(a\zeta_\ell + b) \eta_\infty(a\zeta_\ell + b) N_{L/\mathbb{Q}}((a\zeta_\ell + b))^{-s} \\
&= \frac{1}{|\mathcal{O}_L^\times : \mathcal{O}_F^\times|} \sum'_{(a,b) \in \mathcal{O}_F \times \mathcal{O}_F/\mathcal{O}_F^\times} \phi(a, b) N(a\mathbf{z}_{CM} + b) |N(a\mathbf{z}_{CM} + b)|^{-2s} \\
&= \frac{1}{|\mathcal{O}_L^\times : \mathcal{O}_F^\times|} E(\phi, \mathbf{z}_{CM}, s).
\end{aligned}$$

## 7. A FUNCTION PERIODIC MODULO $\mathfrak{m} \cap \mathcal{O}_F$

Define the function  $f : \mathbb{H}^d \rightarrow \mathbb{C}$  by

$$f(z) = \sum_{g \in \mathfrak{m} \cap \mathcal{O}_F} N(z + g) |N(z + g)|^{-2s}, \quad \operatorname{Re}(s) > \frac{3}{2}.$$

We will compute the Fourier expansion of  $f(z)$ .

Let  $\mathfrak{t}$  be a lattice in  $\mathbb{R}^d$ . Then

$$\mathfrak{t} = \mathbb{Z}r_1 + \cdots + \mathbb{Z}r_d$$

for linearly independent vectors  $r_1, \dots, r_d \in \mathbb{R}^d$ . The set

$$P = \{x \in \mathbb{R}^d : x = \sum_{j=1}^d t_j r_j, 0 \leq t_j \leq 1\}$$

is a fundamental parallelopete for the lattice  $\mathfrak{t}$ . The dual lattice is defined by

$$\mathfrak{t}^* = \{a \in \mathbb{R}^d : T(ax) \in \mathbb{Z} \text{ for all } x \in \mathfrak{t}\},$$

where

$$T = \sum_{j=1}^d a_j x_j$$

is the trace.

Let  $V \subset \mathbb{R}^d$  be an open connected domain. Then

$$D = \{z \in \mathbb{C}^d : y \in V\}$$

is the tube domain corresponding to  $V$  (here  $z = x + iy$ ).

Define the integral

$$h(y; s-1, s) = \int_{\mathbb{R}} (1-it) |1-it|^{-2s} e^{-ity} dt, \quad \operatorname{Re}(s) > 2, \quad y \in \mathbb{R}.$$

By [F, Lemma 4.4] we know that  $h(y; s-1, s)$  has a meromorphic continuation to  $\mathbb{C}$  (entire if  $y \neq 0$ ). Furthermore, by [F, Lemma 4.4 (a)] we have the following special value

$$h(0; s-1, s) = \frac{2\pi\Gamma(2s-2)}{\Gamma(s-1)\Gamma(s)} 2^{2-2s}. \quad (7.1)$$

Let  $P$  be a fundamental parallelopete of the lattice  $\mathfrak{m} \cap \mathcal{O}_F$ . The volume of  $P$  is given by the formula

$$\operatorname{vol}(P) = \sqrt{|\operatorname{disc}(\mathfrak{m} \cap \mathcal{O}_F)|} = D_F^{1/2} N_{F/\mathbb{Q}}(\mathfrak{m} \cap \mathcal{O}_F)$$

where  $\operatorname{disc}(\mathfrak{m} \cap \mathcal{O}_F)$  is the discriminant of the ideal  $\mathfrak{m} \cap \mathcal{O}_F$  and  $D_F$  is the absolute discriminant of  $F$ .

Now,

$$\mathbb{H}^d = \{z \in \mathbb{C}^d : y \in \mathbb{R}_{>0}^d\}$$

is the tube domain corresponding to the open connected domain  $\mathbb{R}_{>0}^d$ . We know that  $f(z)$  is holomorphic on  $\mathbb{H}^d$  and that

$$f(z+g) = f(z)$$

for all  $g \in \mathfrak{m} \cap \mathcal{O}_F$ . Then by [F, Proposition 4.5],  $f(z)$  has the Fourier expansion

$$f(z) = N_{F/\mathbb{Q}}(\mathfrak{m} \cap \mathcal{O}_F) D_F^{1/2} e^{\pi i d/2} \sum_{g \in (\mathfrak{m} \cap \mathcal{O}_F)^*} h_g(y, s) \exp\{2\pi i T(gx)\} \quad (7.2)$$

where

$$h_g(y, s) = N(y)^{2-2s} \prod_{j=1}^d h(2\pi\tau_j(gy); s-1, s).$$

## 8. PROOF OF THEOREM 1.3

In this section we compute the Fourier expansion of  $E(\phi, z, s)$ . We begin with the decomposition

$$\begin{aligned} E(\phi, z, s) &= \sum'_{b \in \mathcal{O}_F / \mathcal{O}_F^\times} \phi(0, b) N(b) N_{F/\mathbb{Q}}((b))^{-2s} \\ &+ \sum'_{a \in \mathcal{O}_F / \mathcal{O}_F^\times} \sum_{b \in \mathcal{O}_F} \phi(a, b) N(az + b) |N(az + b)|^{-2s}. \end{aligned}$$

Let  $\{b_1, \dots, b_N\}$  be a complete set of coset representatives of  $\mathcal{O}_F / \mathfrak{m} \cap \mathcal{O}_F$ . Using that  $\phi$  is periodic modulo  $\mathfrak{m} \cap \mathcal{O}_F$  we find that

$$\begin{aligned} &\sum_{b \in \mathcal{O}_F} \phi(a, b) N(az + b) |N(az + b)|^{-2s} \\ &= \sum_{l=1}^N \phi(a, b_l) \sum_{b \equiv b_l \pmod{\mathfrak{m} \cap \mathcal{O}_F}} N(az + b) |N(az + b)|^{-2s} \\ &= \sum_{l=1}^N \phi(a, b_l) \sum_{g \in \mathfrak{m} \cap \mathcal{O}_F} N(az + b_l + g) |N(az + b_l + g)|^{-2s} \end{aligned}$$

Let  $z \mapsto az + b_l$  in (7.2). Then

$$\begin{aligned} &\sum_{g \in \mathfrak{m} \cap \mathcal{O}_F} N(az + b_l + g) |N(az + b_l + g)|^{-2s} \\ &= N_{F/\mathbb{Q}}(\mathfrak{m} \cap \mathcal{O}_F) D_F^{1/2} e^{\pi i d/2} \sum_{g \in (\mathfrak{m} \cap \mathcal{O}_F)^*} h_g(ay, s) \exp\{2\pi i T(g(ax + b_l))\} \end{aligned}$$

where

$$h_g(ay, s) = N(a)^{2-2s} N(y)^{2-2s} \prod_{j=1}^d h(2\pi\tau_j(agy); s-1, s).$$

Using the special value (7.1) we obtain

$$h_0(ay, s) = N(a)^{2-2s} N(y)^{2-2s} \left\{ \frac{2\pi\Gamma(2s-2)2^{2-2s}}{\Gamma(s-1)\Gamma(s)} \right\}^d.$$

Then from the formula for  $h_0(ay, s)$  we find that

$$\begin{aligned}
& \sum'_{a \in \mathcal{O}_F / \mathcal{O}_F^\times} \sum_{l=1}^N \phi(a, b_l) \sum_{g \in \mathfrak{m} \cap \mathcal{O}_F} N(az + b_l + g) |N(az + b_l + g)|^{-2s} \\
&= N_{F/\mathbb{Q}}(\mathfrak{m} \cap \mathcal{O}_F) D_F^{1/2} e^{\pi i d/2} N(y)^{2-2s} \left\{ \frac{2\pi \Gamma(2s-2) 2^{2-2s}}{\Gamma(s-1)\Gamma(s)} \right\}^d \\
&\quad \times \sum_{l=1}^N \sum'_{a \in \mathcal{O}_F / \mathcal{O}_F^\times} \phi(a, b_l) N_{F/\mathbb{Q}}((a))^{-(2s-2)} \\
&+ N_{F/\mathbb{Q}}(\mathfrak{m} \cap \mathcal{O}_F) D_F^{1/2} e^{\pi i d/2} \\
&\quad \times \sum_{l=1}^N \sum'_{a \in \mathcal{O}_F / \mathcal{O}_F^\times} \phi(a, b_l) \sum'_{g \in (\mathfrak{m} \cap \mathcal{O}_F)^*} h_g(ay, s) \exp\{2\pi i T(g(ax + b_l))\}.
\end{aligned}$$

Let  $\{a_1, \dots, a_N\}$  be (another) complete set of coset representatives of  $\mathcal{O}_F / \mathfrak{m} \cap \mathcal{O}_F$ , and let  $a(\mathfrak{m} \cap \mathcal{O}_F)$  denote the sum over a complete set of representatives  $a \in \mathcal{O}_F$  with respect to the equivalence relation  $a \sim a'$  modulo  $\mathfrak{m} \cap \mathcal{O}_F$  if and only if there exists a unit  $\epsilon \equiv 1 \pmod{\mathfrak{m} \cap \mathcal{O}_F}$  such that  $a = \epsilon a'$ . Then using that  $\phi$  is periodic modulo  $\mathfrak{m} \cap \mathcal{O}_F$  we have

$$\begin{aligned}
& \sum_{l=1}^N \sum'_{a \in \mathcal{O}_F / \mathcal{O}_F^\times} \phi(a, b_l) \sum'_{g \in (\mathfrak{m} \cap \mathcal{O}_F)^*} h_g(ay, s) \exp\{2\pi i T(g(ax + b_l))\} \\
&= \sum_{k=1}^N \sum_{l=1}^N \phi(a_k, b_l) \sum'_{\substack{a \equiv a_k \pmod{\mathfrak{m} \cap \mathcal{O}_F} \\ a(\mathfrak{m} \cap \mathcal{O}_F)}} \sum'_{g \in (\mathfrak{m} \cap \mathcal{O}_F)^*} h_g(ay, s) \exp\{2\pi i T(g(ax + b_l))\}.
\end{aligned}$$

Using the definition of  $h_g$ , additivity of the trace  $T$ , and combining terms with fixed  $ag$ , we obtain

$$\begin{aligned}
& \sum'_{\substack{a \equiv a_k \pmod{\mathfrak{m} \cap \mathcal{O}_F} \\ a(\mathfrak{m} \cap \mathcal{O}_F)}} \sum'_{g \in (\mathfrak{m} \cap \mathcal{O}_F)^*} h_g(ay, s) \exp\{2\pi i T(g(ax + b_l))\} \\
&= N(y)^{2-2s} \sum'_{\tilde{g} \in (\mathfrak{m} \cap \mathcal{O}_F)^*} \sigma_{\tilde{g}}(k, l, y, s) \exp\{2\pi i T(\tilde{g}x)\}
\end{aligned}$$

where the sum of divisors  $\sigma_{\tilde{g}}(k, l, y, s)$  is defined by

$$\sigma_{\tilde{g}}(k, l, y, s) = \sum_{\substack{\tilde{g}=ag \\ g \in (\mathfrak{m} \cap \mathcal{O}_F)^* \\ a \equiv a_k \pmod{\mathfrak{m} \cap \mathcal{O}_F} \\ a(\mathfrak{m} \cap \mathcal{O}_F)}} \prod_{j=1}^d h(2\pi \tau_j(agy); s-1, s) N(a)^{2-2s} \exp\{2\pi i T(gb_l)\}.$$

Define the  $L$ -functions

$$L(\phi, s) = \sum'_{b \in \mathcal{O}_F / \mathcal{O}_F^\times} \phi(0, b) N(b) N_{F/\mathbb{Q}}((b))^{-s},$$

and

$$L(\phi_l, s) = \sum'_{a \in \mathcal{O}_F / \mathcal{O}_F^\times} \phi(a, b_l) N_{F/\mathbb{Q}}((a))^{-s}, \quad l = 1, \dots, N,$$

which are absolutely convergent for  $\operatorname{Re}(s)$  sufficiently large. By combining the preceding analysis we obtain the Fourier expansion of  $E(\phi, z, s)$  in the theorem.

## 9. PROOF OF COROLLARY 1.5

Let  $s = 1$  in Corollary 1.4 to obtain the following formula for the central value  $L(\eta_A, 1)$ ,

$$\begin{aligned} |\mathcal{O}_L^\times : \mathcal{O}_F^\times| L(\eta_A, 1) &= L(\phi, 2) + N_{F/\mathbb{Q}}(\mathfrak{m} \cap \mathcal{O}_F) D_F^{1/2} e^{\pi i d/2} (2\pi)^d \sum_{l=1}^N L(\phi_l, 0) \\ &+ N_{F/\mathbb{Q}}(\mathfrak{m} \cap \mathcal{O}_F) D_F^{1/2} e^{\pi i d/2} \\ &\times \sum_{k=1}^N \sum_{l=1}^N \phi(a_k, b_l) \sum'_{\tilde{g} \in (\mathfrak{m} \cap \mathcal{O}_F)^*} \sigma_{\tilde{g}}(k, l, \mathbf{y}_{CM}, 1) \exp\{2\pi i T(\tilde{g} \mathbf{x}_{CM})\} \end{aligned}$$

where

$$\sigma_{\tilde{g}}(k, l, \mathbf{y}_{CM}, 1) = \sum_{\substack{\tilde{g}=ag \\ g \in (\mathfrak{m} \cap \mathcal{O}_F)^* \\ a \equiv a_k \pmod{\mathfrak{m} \cap \mathcal{O}_F} \\ a \in (\mathfrak{m} \cap \mathcal{O}_F)}} \prod_{j=1}^d h(2\pi \tau_j(ag \mathbf{y}_{CM}); 0, 1) \exp\{2\pi i T(gb_l)\}.$$

It remains to compute the integrals  $h(2\pi \tau_j(ag \mathbf{y}_{CM}); 0, 1)$  appearing in the sum of divisors  $\sigma_{\tilde{g}}(k, l, \mathbf{y}_{CM}, 1)$ .

**Lemma 9.1.** *With notation as above we have*

$$\prod_{j=1}^d h(2\pi \tau_j(ag \mathbf{y}); 0, 1) = \begin{cases} 0, & \text{if } \tau_j(ag) > 0 \text{ for some } j, \\ (2\pi)^d \exp\{2\pi T(ag \mathbf{y})\}, & \text{if } \tau_j(ag) < 0 \text{ for } j = 1, \dots, d. \end{cases}$$

*Proof.* By [F, Lemma 4.4 (b)], if  $\tau_j(ag) > 0$  we have

$$h(2\pi \tau_j(ag \mathbf{y}); 0, 1) = \exp\{-2\pi \tau_j(ag \mathbf{y})\} P_{0,1}(2\pi \tau_j(ag \mathbf{y}))$$

where  $P_{0,1}(x) = 0$ , and if  $\tau_j(ag) < 0$  we have

$$h(2\pi \tau_j(ag \mathbf{y}); 0, 1) = h(-2\pi \tau_j(ag \mathbf{y}); 1, 0) = \exp\{2\pi \tau_j(ag \mathbf{y})\} P_{1,0}(-2\pi \tau_j(ag \mathbf{y}))$$

where  $P_{1,0}(x) = 2\pi$ . □

## 10. PROOF OF COROLLARY 1.6

Recall that the conductor  $\mathfrak{f}$  of  $\eta_A$  is the smallest divisor  $\mathfrak{f}|\mathfrak{m}$  such that  $\eta_A$  is the restriction of a Grössencharakter modulo  $\mathfrak{f}$ . By [S3, Proposition 3.3] the conductor of  $\eta_1$  is

$$\mathfrak{f} = \mathfrak{m} = 2(1 - \zeta_5).$$

We will need the following lemma.

**Lemma 10.1.** *We have (i)  $\mathfrak{f} \cap \mathcal{O}_F = 2\sqrt{5}\mathcal{O}_F$  and (ii)  $2\sqrt{5}\mathcal{O}_L \subset \mathfrak{f}$ .*

*Proof.* (i) By [N, Lemma 10.1], we have  $5\mathcal{O}_L = (1 - \zeta_5)^4\mathcal{O}_L$ , where  $(1 - \zeta_5)\mathcal{O}_L$  is *prime*. Now,  $(\sqrt{5}\mathcal{O}_L)^2 = 5\mathcal{O}_L = (1 - \zeta_5)^4\mathcal{O}_L$ . Consider the prime decomposition  $\sqrt{5}\mathcal{O}_L = P_1^{e_1} \cdots P_n^{e_n}$ . Then  $(\sqrt{5}\mathcal{O}_L)^2 = P_1^{2e_1} \cdots P_n^{2e_n}$ , so by uniqueness of prime decompositions in Dedekind domains we have  $P_1 = (1 - \zeta_5)\mathcal{O}_L$ ,  $e_1 = 2$ , and  $e_i = 0$  for  $i = 2, \dots, n$ . In particular,  $\sqrt{5}\mathcal{O}_L = (1 - \zeta_5)^2\mathcal{O}_L$ , which implies that the prime  $(1 - \zeta_5)\mathcal{O}_L$  lies above  $\sqrt{5}\mathcal{O}_F$  if and only if  $(1 - \zeta_5)\mathcal{O}_L \cap \mathcal{O}_F = \sqrt{5}\mathcal{O}_F$ , which implies that  $\mathfrak{f} \cap \mathcal{O}_F = 2\sqrt{5}\mathcal{O}_F$ .

(ii) From the proof of (i) we have  $2\sqrt{5}\mathcal{O}_L = 2(1 - \zeta_5)^2\mathcal{O}_L \subset 2(1 - \zeta_5)\mathcal{O}_L = \mathfrak{f}$ .  $\square$

It follows from Lemma 10.1 (i) that  $\mathfrak{m} \cap \mathcal{O}_F = 2\sqrt{5}\mathcal{O}_F$ . Furthermore, it follows from Lemma 10.1 (ii) and an argument similar to that in Lemma 3.1 that  $\phi : \mathcal{O}_F \times \mathcal{O}_F \rightarrow \mathbb{C}$  is periodic modulo  $2\sqrt{5}\mathcal{O}_F$  in each argument.

Now, since  $F = \mathbb{Q}(\sqrt{5})$  we have

$$N_{F/\mathbb{Q}}(\mathfrak{m} \cap \mathcal{O}_F) = N_{F/\mathbb{Q}}(2\sqrt{5}\mathcal{O}_F) = \left| \tau_1(2\sqrt{5}\mathcal{O}_F)\tau_2(2\sqrt{5}\mathcal{O}_F) \right| = 20$$

and  $D_F = 5$ . From Corollary 1.5 with  $A = 1$ ,  $d = 2$ ,  $\mathbf{z}_{CM} = (\zeta_5, \zeta_5^2)$ ,  $\mathfrak{m} \cap \mathcal{O}_F = 2\sqrt{5}\mathcal{O}_F$ , and  $N_{F/\mathbb{Q}}(\mathfrak{m} \cap \mathcal{O}_F)D_F^{1/2} = 20\sqrt{5}$ , we obtain the formula in the corollary.

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