

THE DISTRIBUTION OF ZEROS OF EPSTEIN ZETA FUNCTIONS OVER GL_n

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ABSTRACT. We study the distribution of the nontrivial zeros of ideal class zeta functions associated to elements in the symmetric space of GL_n over a number field. We establish asymptotics for the number of nontrivial zeros up to height T , and asymptotics for the distribution of the nontrivial zeros with respect to the critical line. We combine these results to study the mean value of the real parts of the nontrivial zeros.

1. INTRODUCTION

1. The study of the distribution of the zeros of Epstein zeta functions has a long and interesting history (see e.g. [BG, BH, BM, DH, E, H, St]). In this paper we study the distribution of the nontrivial zeros of ideal class zeta functions associated to elements in the symmetric space of GL_n over a number field K (see Definition 1.1). These zeta functions have been studied extensively by Terras in [T1, T2, T3, T4]. They generalize the classical Epstein zeta function of a positive definite quadratic form. Furthermore, when $n = 2$ they play an important role in the study of the spectral theory of the Laplacian through their relationship with SL_2 -Eisenstein series (see [M]). We will use a method of Levinson and Montgomery [LM] to establish asymptotics for the number of nontrivial zeros up to height T , and asymptotics for the distribution of the nontrivial zeros with respect to the critical line. We then combine these results to study the mean value of the real parts of the nontrivial zeros. Our results generalize those of Steuding [S] for $K = \mathbb{Q}$.

2. We now define the zeta functions to be studied in this paper (see also [T4]). Let n be a positive integer and K be a number field of degree $N = r_1 + 2r_2$ over \mathbb{Q} . The symmetric space \mathcal{Q}_K^n of positive n -forms over K consists of $r_1 + r_2$ -tuples $Q = (Q^{(1)}, \dots, Q^{(r_1+r_2)})$ of positive definite $n \times n$ matrices such that the first r_1 are real symmetric and the second r_2 are complex Hermitian. The determinant one subspace \mathcal{SQ}_K^n consists of Q in \mathcal{Q}_K^n such that $\det(Q^{(j)}) = 1$ for $j = 1, \dots, r_1 + r_2$. There is an identification

$$\mathcal{Q}_K^n \longleftrightarrow (\mathrm{GL}_n(\mathbb{R})/\mathrm{O}(n))^{r_1} \times (\mathrm{GL}_n(\mathbb{C})/\mathrm{U}(n))^{r_2}$$

which realizes \mathcal{Q}_K^n as products of the rank $n - 1$ symmetric spaces $\mathrm{GL}_n(\mathbb{R})/\mathrm{O}(n)$ and $\mathrm{GL}_n(\mathbb{C})/\mathrm{U}(n)$, respectively. The group $\mathrm{GL}_n(K)$ acts on \mathcal{Q}_K^n by

$$Q[A] = \left(\overline{A^{(1)}}^t Q^{(1)} A^{(1)}, \dots, \overline{A^{(r_1+r_2)}}^t Q^{(r_1+r_2)} A^{(r_1+r_2)} \right) \quad \text{for } A \in \mathrm{GL}_n(K),$$

where $A^{(j)}$ means the j -th embedding of K is applied to each entry of A .

Let \mathcal{O}_K be the ring of integers and U_K be the group of units of K . Given an integral ideal \mathfrak{a} in K , let $[\mathfrak{a}]$ be its ideal class and $N_{K/\mathbb{Q}}(\mathfrak{a}) = |\mathcal{O}_K : \mathfrak{a}|$ be its norm. Two elements a and a' in \mathfrak{a}^n are said to be associate modulo U_K if there exists a unit ϵ in U_K such that

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$a = \epsilon a'$. Let $\mathcal{R}(\mathfrak{a}^n)$ be a complete set of nonzero, non-associate elements in \mathfrak{a}^n . Finally, let S_∞ be a collection of $r_1 + r_2$ infinite places ν of K , and let

$$N_\nu = \begin{cases} 1 & \text{if } \nu \text{ is real,} \\ 2 & \text{if } \nu \text{ is complex.} \end{cases}$$

Definition 1.1. *The ideal class zeta function associated to a positive n -form Q in \mathcal{Q}_K^n is defined by*

$$\zeta_n(Q, [\mathfrak{a}], s) = N_{K/\mathbb{Q}}(\mathfrak{a})^{2s} \sum_{a \in \mathcal{R}(\mathfrak{a}^n)} N(Q[a])^{-s}, \quad \operatorname{Re}(s) > \frac{n}{2},$$

where

$$N(Q[a]) = \prod_{\nu \in S_\infty} Q^{(\nu)}[a^{(\nu)}]^{N_\nu}.$$

Remarks. (a) The factor $N_{K/\mathbb{Q}}(\mathfrak{a})^{2s}$ insures that $\zeta_n(Q, [\mathfrak{a}], s)$ depends only on the ideal class $[\mathfrak{a}]$.

(b) It is a consequence of the definition that $\zeta_n(Q, [\mathfrak{a}], s)$ is automorphic with respect to the action of $\operatorname{GL}_n(\mathcal{O}_K)$ on \mathcal{Q}_K^n , i.e.

$$\zeta_n(Q[A], [\mathfrak{a}], s) = \zeta_n(Q, [\mathfrak{a}], s) \quad \text{for all } A \in \operatorname{GL}_n(\mathcal{O}_K).$$

(c) When $K = \mathbb{Q}$ the zeta function $\zeta_n(Q, [\mathfrak{a}], s)$ equals the classical Epstein zeta function of a positive definite real symmetric matrix Q ,

$$\zeta_n(Q, [\mathbb{Z}], s) = \frac{1}{2} \sum_{a \in \mathbb{Z}^n \setminus \{0\}} Q[a]^{-s}, \quad \operatorname{Re}(s) > \frac{n}{2}.$$

(d) The zeta functions $\zeta_2(Q, [\mathfrak{a}], s)$ can be used to compute the scattering matrix and scattering determinant associated to the functional equation satisfied by the SL_2 -Eisenstein series corresponding to the cusps of the non-compact, finite volume quotient $X = \operatorname{SL}_2(\mathcal{O}_K) \backslash \mathcal{S}\mathcal{Q}_K^2$ (see [M]). One can combine this information with the Selberg trace formula for X to establish Weyl's law for the function $N(T)$ which counts the number of eigenvalues $\lambda \leq T$ corresponding to cuspidal eigenfunctions for the Laplacian Δ in $L^2(X)$ (see [R, Sa1, Sa2]).

3. We will make repeated use of the functional equation for $\zeta_n(Q, [\mathfrak{a}], s)$ (for the proof see [M]).

Theorem 1.2. *Let $A = 2^{-r_2} d_K^{\frac{1}{2}} \pi^{-\frac{N}{2}}$, where d_K is the absolute value of the discriminant of K , and let \mathfrak{a}^* be the dual of \mathfrak{a} . Then $\zeta_n(Q, [\mathfrak{a}], s)$ is meromorphic on \mathbb{C} with a simple pole at $s = \frac{n}{2}$, and*

$$\begin{aligned} A^{2s} \Gamma(s)^{r_1} \Gamma(2s)^{r_2} \zeta_n(Q, [\mathfrak{a}], s) = & \quad (1.1) \\ \|\|Q\|\|^{-\frac{1}{2}} A^{2(\frac{n}{2}-s)} \Gamma\left(\frac{n}{2}-s\right)^{r_1} \Gamma\left(2\left(\frac{n}{2}-s\right)\right)^{r_2} \zeta_n\left(Q^{-1}, [\mathfrak{a}^*], \frac{n}{2}-s\right), \end{aligned}$$

where

$$\|\|Q\|\| = \prod_{\nu \in S_\infty} \left| \det(Q^{(\nu)}) \right|^{N_\nu}$$

is the norm of the determinant of Q .

It follows from the functional equation (1.1) that $\zeta_n(Q, [\mathbf{a}], s)$ has zeros at $m = -1, -2, \dots$. We refer to these as trivial zeros and to all other zeros $\rho = \beta + i\gamma$ as *nontrivial* zeros.

4. We now state our main results. Let

$$N(Q, [\mathbf{a}], T) = |\{\rho = \beta + i\gamma : |\gamma| \leq T\}|$$

count the number of nontrivial zeros ρ with $|\gamma| \leq T$. Here the zeros are counted according to multiplicities. Let

$$m(Q, [\mathbf{a}]) = \min \left\{ \frac{N(Q[\mathbf{a}])}{N_{K/\mathbb{Q}}(\mathbf{a})^2} : a \in \mathcal{R}(\mathfrak{a}^n) \right\}$$

and

$$M(Q, [\mathbf{a}]) = \left| \left\{ a \in \mathcal{R}(\mathfrak{a}^n) : \frac{N(Q[\mathbf{a}])}{N_{K/\mathbb{Q}}(\mathbf{a})^2} = m(Q, [\mathbf{a}]) \right\} \right|.$$

It is not difficult to show that $m(Q, [\mathbf{a}])$ and $M(Q, [\mathbf{a}])$ depend only on the ideal class $[\mathbf{a}]$.

In the following theorem we establish asymptotics for the number of nontrivial zeros up to height T , and asymptotics for the distribution of the nontrivial zeros with respect to the critical line.

Theorem 1.3. *As $T \rightarrow \infty$,*

$$N(Q, [\mathbf{a}], T) = \frac{2NT}{\pi} \log \left\{ \frac{T}{\pi e \sqrt{m(Q, [\mathbf{a}])m(Q^{-1}, [\mathbf{a}^*])}} \right\} + O(\log T) \quad (1.2)$$

and

$$\sum_{|\gamma| \leq T} \left(\beta - \frac{n}{4} \right) = -\frac{NT}{\pi} \log \left\{ \|Q\|^{\frac{1}{2N}} \left(\frac{M(Q, [\mathbf{a}])}{M(Q^{-1}, [\mathbf{a}^*])} \right)^{\frac{1}{N}} \left(\frac{m(Q^{-1}, [\mathbf{a}^*])}{m(Q, [\mathbf{a}])} \right)^{\frac{n}{4}} \right\} + O(\log T). \quad (1.3)$$

Equation (1.3) measures, in some sense, the asymmetries in the distribution of the nontrivial zeros of $\zeta_n(Q, [\mathbf{a}], s)$ with respect to the critical line. To see this, consider the constant

$$\Sigma(Q, [\mathbf{a}]) = \log \left\{ \|Q\|^{\frac{1}{2N}} \left(\frac{M(Q, [\mathbf{a}])}{M(Q^{-1}, [\mathbf{a}^*])} \right)^{\frac{1}{N}} \left(\frac{m(Q^{-1}, [\mathbf{a}^*])}{m(Q, [\mathbf{a}])} \right)^{\frac{n}{4}} \right\}.$$

If $K = \mathbb{Q}$ and $n = 2$ so that Q corresponds to a binary quadratic form, then $\zeta_2(Q, [\mathbb{Z}], s)$ is a constant multiple of $\zeta_2(Q^{-1}, [\mathbb{Z}], s)$ which implies via the functional equation (1.1) that there is a bijection between the zeros ρ and $1 - \rho$. It follows immediately that $\Sigma(Q, [\mathbb{Z}]) = 0$. Note, however, that it is possible that there are configurations of a few asymmetrically distributed zeros such that $\Sigma(Q, [\mathbf{a}]) = 0$. It is not difficult to construct examples with $\Sigma(Q, [\mathbb{Z}]) \neq 0$ (see [S]).

By combining equations (1.2) and (1.3) we find that the *mean* value of the real parts of the nontrivial zeros of $\zeta_n(Q, [\mathbf{a}], s)$ exists and equals $\frac{n}{4}$, and thus the possible asymmetries in the distribution of the nontrivial zeros are small when compared with the total number of zeros.

Corollary 1.4. *As $T \rightarrow \infty$,*

$$\frac{1}{N(Q, [\mathbf{a}], T)} \sum_{|\gamma| \leq T} \beta = \frac{n}{4} + O\left(\frac{1}{\log T}\right).$$

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2. PROOF OF THEOREM 1.3

The proof of Theorem 1.3 is based on a method of Levinson and Montgomery [LM] used to study the distribution of the zeros of the derivatives of the Riemann zeta function. See also the elaboration on this method in [S].

We will use the following lemma due to Littlewood [Li].

Lemma 2.1. *Let $A < B$ and let $f(s)$ be analytic on $\mathcal{R} = \{s \in \mathbb{C} : A \leq \sigma := \operatorname{Re}(s) \leq B, |t| \leq T\}$. Suppose that $f(s)$ does not vanish on the right edge $\sigma = B$ of \mathcal{R} . Let \mathcal{R}' be \mathcal{R} minus the union of the horizontal cuts from the zeros of f in \mathcal{R} to the left edge of \mathcal{R} , and choose a single-valued branch of $\log f(s)$ in the interior of \mathcal{R}' . Denote by $v(\sigma, T)$ the number of zeros $\rho = \beta + i\gamma$ of $f(s)$ inside \mathcal{R} with $\beta > \sigma$ including zeros with $\gamma = T$ but not those with $\gamma = -T$. Then*

$$\int_{\partial\mathcal{R}} \log f(s) ds = -2\pi i \int_A^B v(\sigma, T) d\sigma.$$

See [T], section 9.9 for a proof.

Define

$$Z_n(Q, [\mathbf{a}], s) = \frac{m(Q, [\mathbf{a}])^s}{M(Q, [\mathbf{a}])} \zeta_n(Q, [\mathbf{a}], s). \quad (2.1)$$

Then $Z_n(Q, [\mathbf{a}], s)$ has the same zeros as $\zeta_n(Q, [\mathbf{a}], s)$, and for $\sigma > \frac{n}{2}$,

$$Z_n(Q, [\mathbf{a}], s) = 1 + \frac{1}{M(Q, [\mathbf{a}])} \sum_{\substack{a \in \mathcal{R}(\mathfrak{a}^n) \\ \frac{N(Q[a])}{N_{K/\mathbb{Q}}(\mathfrak{a})^2} > m(Q, [\mathbf{a}])}} \left(\frac{m(Q, [\mathbf{a}])}{N_{K/\mathbb{Q}}(\mathfrak{a})^{-2} N(Q[a])} \right)^s.$$

Let

$$m_2(Q, [\mathbf{a}]) = \inf \left\{ a \in \mathcal{R}(\mathfrak{a}^n) : \frac{N(Q[a])}{N_{K/\mathbb{Q}}(\mathfrak{a})^2} > m(Q, [\mathbf{a}]) \right\}.$$

Again, it is not difficult to show that $m_2(Q, [\mathbf{a}])$ depends only on the ideal class $[\mathbf{a}]$. Then

$$Z_n(Q, [\mathbf{a}], s) = 1 + C(Q, [\mathbf{a}]) \lambda^{-s} + o(\lambda^{-\sigma}) \quad \text{as } \sigma \rightarrow \infty, \quad (2.2)$$

where $C(Q, [\mathbf{a}]) > 0$ is a constant, and

$$\lambda = \frac{m_2(Q, [\mathbf{a}])}{m(Q, [\mathbf{a}])} > 1.$$

Consequently,

$$Z_n(Q, [\mathbf{a}], s) \rightarrow 1 \quad \text{as } \sigma \rightarrow \infty,$$

so there exists a zero-free region for $\zeta_n(Q, [\mathbf{a}], s)$ for σ sufficiently large. Similarly, there exists a zero-free region for $\zeta_n(Q^{-1}, [\mathbf{a}^*], s)$ for σ sufficiently large. It follows from the functional equation (1.1) that there exists a constant $B > 0$ such that $\zeta_n(Q, [\mathbf{a}], s)$ and $\zeta_n(Q^{-1}, [\mathbf{a}^*], s)$ have no nontrivial zeros outside of the strip $-B < \sigma < B$.

Define the function

$$f(s) = \left(s - \frac{n}{2} \right) Z_n(Q, [\mathbf{a}], s).$$

Then $f(s)$ is entire and has the same zeros as $\zeta_n(Q, [\mathbf{a}], s)$. Let $b > B$ be sufficiently large. Form the rectangle \mathcal{R} with vertices $-b \pm iT$, $B \pm iT$, and assume that $f(s)$ does not vanish on $\partial\mathcal{R}$. In what follows let $\rho = \beta + i\gamma$ denote *any* zero of $\zeta_n(Q, [\mathbf{a}], s)$ (for any T there are at most b trivial zeros of $\zeta_n(Q, [\mathbf{a}], s)$ inside \mathcal{R} and thus their contribution is negligible).

We now apply Lemma 2.1 to the function $f(s)$. First, observe that

$$\int_{-b}^B v(\sigma, T) d\sigma$$

is the total length of the horizontal cuts from the zeros of $f(s)$ in \mathcal{R} to the left edge of \mathcal{R} . Then

$$\int_{-b}^B v(\sigma, T) d\sigma = \sum_{|\gamma| \leq T} (\beta + b),$$

so that by Lemma 2.1,

$$i \int_{\partial\mathcal{R}} \log f(s) ds = 2\pi \sum_{|\gamma| \leq T} (\beta + b). \quad (2.3)$$

Next, write

$$i \int_{\partial\mathcal{R}} \log f(s) ds = i \int_{\partial\mathcal{R}} \log \left(s - \frac{n}{2} \right) ds + i \int_{\partial\mathcal{R}} \log Z_n(Q, [\mathbf{a}], s) ds. \quad (2.4)$$

By Lemma 2.1,

$$i \int_{\partial\mathcal{R}} \log \left(s - \frac{n}{2} \right) ds = 2\pi \left(\frac{n}{2} + b \right). \quad (2.5)$$

Furthermore (see also [T], section 9.9, equation (9.9.2)),

$$\begin{aligned} i \int_{\partial\mathcal{R}} \log Z_n(Q, [\mathbf{a}], s) ds = & \quad (2.6) \\ & \int_{-T}^T \log Z_n(Q, [\mathbf{a}], -b + it) dt - \int_{-T}^T \log Z_n(Q, [\mathbf{a}], B + it) dt \\ & + i \int_{-b}^B \{ \log Z_n(Q, [\mathbf{a}], \sigma - iT) - \log Z_n(Q, [\mathbf{a}], \sigma + iT) \} d\sigma. \end{aligned}$$

By equations (2.3), (2.4) and (2.5), the right hand side of (2.6) is real. Expand the right hand side of (2.6) and let the imaginary part equal zero to express equation (2.3) in the form

$$\begin{aligned} 2\pi \sum_{|\gamma| \leq T} (\beta + b) = & \int_{-T}^T \log |Z_n(Q, [\mathbf{a}], -b + it)| dt - \int_{-T}^T \log |Z_n(Q, [\mathbf{a}], B + it)| dt \\ & - \int_{-b}^B \arg Z_n(Q, [\mathbf{a}], \sigma - iT) d\sigma + \int_{-b}^B \arg Z_n(Q, [\mathbf{a}], \sigma + iT) d\sigma \\ & + O(1). \quad (2.7) \end{aligned}$$

From (2.2),

$$\log Z_n(Q, [\mathbf{a}], s) \sim C(Q, [\mathbf{a}]) \lambda^{-s} \quad \text{as } \sigma \rightarrow \infty,$$

thus for sufficiently large B ,

$$\int_{-T}^T \log |Z_n(Q, [\mathbf{a}], B + it)| dt = O(1) \quad (2.8)$$

as $T \rightarrow \infty$.

We now estimate the integral

$$\int_{-T}^T \log |Z_n(Q, [\mathbf{a}], -b + it)| dt.$$

From (2.1),

$$\begin{aligned} \log |Z_n(Q, [\mathbf{a}], -b + it)| = \\ -b \log m(Q, [\mathbf{a}]) - \log M(Q, [\mathbf{a}]) + \log |\zeta_n(Q, [\mathbf{a}], -b + it)|. \end{aligned}$$

Furthermore, from the functional equation (1.1),

$$\begin{aligned} \log |\zeta_n(Q, [\mathbf{a}], -b + it)| = \\ -\frac{1}{2} \log \|Q\| + (n + 4b) \log A \\ + r_1 \left\{ \log \left| \Gamma \left(\frac{n}{2} + b - it \right) \right| - \log |\Gamma(-b + it)| \right\} \\ + r_2 \left\{ \log |\Gamma(n + 2b - 2it)| - \log |\Gamma(-2b + 2it)| \right\} \\ + \log \left| \zeta_n \left(Q^{-1}, [\mathbf{a}^*], \frac{n}{2} + b - it \right) \right|. \end{aligned}$$

By Stirling's formula,

$$\log |\Gamma(\sigma + it)| = \left(\sigma - \frac{1}{2} \right) \log |t| - \frac{\pi}{2} |t| + \frac{1}{2} \log(2\pi) + O(|t|^{-1}), \quad (2.9)$$

for $|t| \geq 1$ and σ bounded. Using (2.9) we obtain, after a calculation, that

$$\begin{aligned} \int_{-T}^T \log |Z_n(Q, [\mathbf{a}], -b + it)| dt = \\ r_1(n + 4b) \int_1^T \log t dt + r_2(n + 4b) \int_1^{2T} \log t dt \\ - T \{ \log \|Q\| - 2(n + 4b) \log A + 2b \log m(Q, [\mathbf{a}]) + 2 \log M(Q, [\mathbf{a}]) \} \\ + \int_{-T}^T \log \left| \zeta_n \left(Q^{-1}, [\mathbf{a}^*], \frac{n}{2} + b - it \right) \right| dt + O(\log T). \end{aligned}$$

From (2.1),

$$\begin{aligned} \int_{-T}^T \log \left| \zeta_n \left(Q^{-1}, [\mathbf{a}^*], \frac{n}{2} + b - it \right) \right| dt = \\ \int_{-T}^T \log \left| Z_n \left(Q^{-1}, [\mathbf{a}^*], \frac{n}{2} + b - it \right) \right| dt \\ + T \{ 2 \log M(Q^{-1}, [\mathbf{a}^*]) - (n + 2b) \log m(Q^{-1}, [\mathbf{a}^*]) \}. \end{aligned} \quad (2.10)$$

The integral on the right hand side of (2.10) can be estimated as in (2.8). Furthermore, it follows from

$$\int_1^T \log t dt = T \log T + T \log(e^{-1}) + O(1)$$

and

$$\log A = \frac{r_1}{2} \log(\pi^{-1}) + r_2 \log(2^{-1} \pi^{-1}) + O(1)$$

that

$$\begin{aligned} r_1(n+4b) \int_1^T \log t dt + r_2(n+4b) \int_1^{2T} \log t dt + 2T(n+4b) \log A = \\ N(n+4b)T \log \frac{T}{\pi e} + O(1). \end{aligned}$$

Combining the preceding analysis yields

$$\begin{aligned} \int_{-T}^T \log |Z_n(Q, [\mathbf{a}], -b + it)| dt = N(n+4b)T \log \frac{T}{\pi e} \\ - T \{ 2 \log M(Q, [\mathbf{a}]) - 2 \log M(Q^{-1}, [\mathbf{a}^*]) + \log \|Q\| \\ + 2b \log m(Q, [\mathbf{a}]) + (n+2b) \log m(Q^{-1}, [\mathbf{a}^*]) \} + O(\log T) \end{aligned} \quad (2.11)$$

as $T \rightarrow \infty$.

We now estimate the integrals

$$\int_{-b}^B \arg Z_n(Q, [\mathbf{a}], \sigma \pm iT) d\sigma.$$

As remarked in [LM], integrals of this type can be estimated by bounding the number of zeros of $\operatorname{Re} \zeta_n(Q, [\mathbf{a}], \sigma + iT)$ using Jensen's formula. Arguing as in [S], we first reduce the problem to bounding the number of zeros. Suppose that $\operatorname{Re} \zeta_n(Q, [\mathbf{a}], \sigma + iT)$ has M zeros for $-b \leq \sigma \leq B$. Divide $[-b, B]$ into at most $M+1$ subintervals in each of which $\operatorname{Re} Z_n(Q, [\mathbf{a}], \sigma + iT)$ is of constant sign. The variation of $\arg Z_n(Q, [\mathbf{a}], \sigma + iT)$ is at most π in each subinterval, thus

$$|\arg Z_n(Q, [\mathbf{a}], \sigma + iT)| \leq (M+1)\pi. \quad (2.12)$$

We now estimate M using an application of Jensen's formula. Let

$$g(z) = \frac{1}{2} \left\{ Z_n(Q, [\mathbf{a}], z + iT) + \overline{Z_n(Q, [\mathbf{a}], \bar{z} + iT)} \right\}.$$

Then $g(\sigma) = \operatorname{Re} Z_n(Q, [\mathbf{a}], \sigma + iT)$. Let $R = b + B$ and choose T so that $T > 2R$. Because $\operatorname{Im}(z + iT) > 0$ in the open disc $|z - B| < T$, $Z_n(Q, [\mathbf{a}], z + iT)$, and hence $g(z)$, is analytic in $|z - B| < T$ (that is, we avoid the pole at $\frac{n}{2}$).

Let $n(r)$ denote the number of zeros of $g(z)$ in the closed disc $|z - B| \leq r$. A simple estimate yields

$$\int_0^{2R} \frac{n(r)}{r} dr \geq n(R) \int_R^{2R} \frac{dr}{r} = n(R) \log 2,$$

and by Jensen's formula,

$$\int_0^{2R} \frac{n(r)}{r} dr = \frac{1}{2\pi} \int_0^{2\pi} \log \left| g \left(B + 2Re^{i\theta} \right) \right| d\theta - \log |g(B)|,$$

so that

$$n(R) \leq \frac{1}{2\pi \log 2} \int_0^{2\pi} \log \left| g \left(B + 2Re^{i\theta} \right) \right| d\theta - \frac{\log |g(B)|}{\log 2}.$$

It follows from (2.2) that $\log |g(B)|$ is bounded. Furthermore, from the Phragmen-Lindelöf theorem we obtain, in any strip of bounded width,

$$\zeta_n(Q, [\mathbf{a}], s) = O \left(|t|^{c(Q, [\mathbf{a}], \sigma)} \right), \quad |t| \rightarrow \infty,$$

for a constant $c(Q, [\mathbf{a}], \sigma) > 0$. The same estimate holds for $g(z)$. Therefore,

$$\int_0^{2\pi} \log \left| g \left(B + 2Re^{i\theta} \right) \right| d\theta = O(\log T),$$

which implies that $n(R) = O(\log T)$. Because $(-b, B)$ is contained in the disc $|z - B| \leq R$, we obtain $M \leq n(R)$. A similar argument can be made for $\arg Z_n(Q, [\mathbf{a}], \sigma - iT)$. Finally, we obtain from (2.12) the estimate

$$\int_{-b}^B |\arg Z_n(Q, [\mathbf{a}], \sigma \pm iT)| d\sigma = O(\log T). \quad (2.13)$$

Combining (2.8), (2.11), and (2.13), we obtain in equation (2.7), after a tedious but straightforward algebraic manipulation,

$$\begin{aligned} \sum_{|\gamma| \leq T} (\beta + b) &= \left(b + \frac{n}{4} \right) \frac{2NT}{\pi} \log \left\{ \frac{T}{\pi e \sqrt{m(Q, [\mathbf{a}]) m(Q^{-1}, [\mathbf{a}^*])}} \right\} \\ &\quad - \frac{NT}{\pi} \log \left\{ \|Q\|^{\frac{1}{2N}} \left(\frac{M(Q, [\mathbf{a}])}{M(Q^{-1}, [\mathbf{a}^*])} \right)^{\frac{1}{N}} \left(\frac{m(Q^{-1}, [\mathbf{a}^*])}{m(Q, [\mathbf{a}])} \right)^{\frac{n}{4}} \right\} \\ &\quad + O(\log T) \end{aligned}$$

as $T \rightarrow \infty$. The assumption that $f(s)$ does not vanish on $\partial\mathcal{R}$ can be removed with an error $O(1)$. Using the identity

$$\sum_{|\gamma| \leq T} (\beta + b) = \left(b + \frac{n}{4} \right) \sum_{|\gamma| \leq T} 1 + \sum_{|\gamma| \leq T} \left(\beta - \frac{n}{4} \right),$$

we obtain equations (1.2) and (1.3). This completes the proof.

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