ASYMPTOTICS FOR SUMS OF CENTRAL VALUES OF CANONICAL HECKE L–SERIES

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Abstract. Let $K = \mathbb{Q}(\sqrt{-D})$ be an imaginary quadratic field of discriminant $-D$ with $D > 3$ and $D \equiv 3 \bmod 4$. Let $\mathcal{O}_K$ be the ring of integers of K, let ε be the quadratic character of K of conductor $\sqrt{-D}\mathcal{O}_K$, and let ψ_k be a Hecke character of K of conductor $\sqrt{-D}\mathcal{O}_K$ satisfying

$$\psi_k(\alpha\mathcal{O}_K) = \varepsilon(\alpha)\alpha^{2k-1}, \quad \text{for} \quad (\alpha\mathcal{O}_K, \sqrt{-D}\mathcal{O}_K) = 1, \quad k \in \mathbb{Z}_{\geq 1}.$$

Let $h(-D)$ be the class number of K, and let Ψ_k be the set of $h(-D)$ Hecke characters of the form ψ_k. If $L(\psi_k, s)$ denotes the L–series of ψ_k, then its central value is $L(\psi_k, k)$. In our main theorem we establish for each even integer $k \geq 2$ an asymptotic formula for the average

$$\frac{1}{h(-D)} \sum_{\psi_k \in \Psi_k} \frac{L(\psi_k, k)}{L((\frac{-D}{\cdot}), 1)}$$

as $D \to \infty$. We then use this formula to prove that there exists an absolute constant $\delta > 0$ such that the number of nonvanishing central values in the family $\{L(\psi_k, k) : \ \psi_k \in \Psi_k\}$ is $\gg D^{\delta}$ as $D \to \infty$.

1. Introduction and statements of results

Formulas which give asymptotics for averages of families of automorphic L–functions are of great interest in analytic number theory. While such formulas are of intrinsic interest, they also have a wide range of important applications, including to the study of lower bounds for L–functions, nonvanishing of critical values and their twists, and Landau-Siegel zeros (see e.g. [Bl, BFH, IS, KMV, RR]). In this paper, we will establish an asymptotic formula for the average of central values of L–series associated to canonical Hecke characters of imaginary quadratic fields. We then use this formula to prove a quantitative nonvanishing theorem for these central values.

The setup for this paper is as follows. Let $K = \mathbb{Q}(\sqrt{-D})$ be an imaginary quadratic field of discriminant $-D$ with $D > 3$ and $D \equiv 3 \bmod 4$. Let $\mathcal{O}_K$ be the ring of integers of K. The group of units $\mathcal{O}_K^{\times} = \{\pm 1\}$. Let $\mathrm{CL}(K)$ be the ideal class group of K and let $\mathrm{CL}_2(K)$ be the subgroup of $\mathrm{CL}(K)$ of exponent 2. The class of an ideal $\mathfrak{a}$ is denoted by $[\mathfrak{a}]$. Let $\mathrm{CL}^{(2)}(K) = \mathrm{CL}(K)/\mathrm{CL}_2(K)$, which is isomorphic to $\mathrm{CL}^2(K)$ under the map $[\mathfrak{a}] \mapsto [\mathfrak{a}^2]$. Unless otherwise stated, all ideals are assumed to be integral. An ideal is primitive if it is not divisible by a rational integer > 1. Any primitive ideal $\mathfrak{a}$ can be written as

$$\mathfrak{a} = \left[a, \frac{b + \sqrt{-D}}{2}\right], \tag{1.1}$$

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where $a = N(\mathfrak{a})$ is the norm of $\mathfrak{a}$ and b is an integer defined modulo $2a$ which satisfies the congruence $b^2 \equiv -D \bmod 4a$. Conversely, given any such pair of numbers a, b, equation (1.1) defines a primitive ideal of norm a.

Let $\varepsilon(n) = (-D/n) = (n/D)$ be the Kronecker symbol associated to K (that the Kronecker symbol equals the Legendre symbol follows from the quadratic reciprocity law). We also view ε as a quadratic character of $(\mathcal{O}_K/\sqrt{-D}\mathcal{O}_K)^\times$ via the isomorphism

$$\mathbb{Z}/D\mathbb{Z} \to \mathcal{O}_K/\sqrt{-D}\mathcal{O}_K.$$

Let ψ_k be a Hecke character of K of conductor $\sqrt{-D}\mathcal{O}_K$ satisfying

$$\psi_k(\alpha\mathcal{O}_K) = \varepsilon(\alpha)\alpha^{2k-1}, \quad \text{for} \quad (\alpha\mathcal{O}_K, \sqrt{-D}\mathcal{O}_K) = 1, \quad k \in \mathbb{Z}_{\geq 1}.$$

Assume in addition that ψ_k satisfies

$$\overline{\psi_k(\mathfrak{a})} = \psi_k(\overline{\mathfrak{a}}) \quad \text{for all ideals } \mathfrak{a} \text{ prime to } \sqrt{-D}\mathcal{O}_K. \tag{1.2}$$

Let $h(-D)$ be the class number of K and let $\mathrm{CL}(K)^\wedge$ be the group of ideal class group characters of K,

$$\xi : \mathrm{CL}(K) \to \overline{\mathbb{Q}}^\times.$$

Let Ψ_k be the set of Hecke characters of the form ψ_k. This set consists of $h(-D)$ characters, and if ψ_k is any one of them,

$$\Psi_k = \{\psi_k\xi : \xi \in \mathrm{CL}(K)^\wedge\}.$$

The L–series of ψ_k is defined by

$$L(\psi_k, s) = \sum_{\mathfrak{a}} \psi_k(\mathfrak{a})N(\mathfrak{a})^{-s}, \quad \text{for} \quad \mathrm{Re}(s) > k + \frac{1}{2},$$

where the sum is over nonzero ideals. It is known that $L(\psi_k, s)$ has an analytic continuation to $\mathbb{C}$ and satisfies a functional equation under $s \mapsto 2k - s$ with root number

$$w(\psi_k) = (-1)^{k-1+\frac{D+1}{4}}$$

(see e.g. [FI, pg. 674]). The central value is $L(\psi_k, k)$.

We now state our main results. Define the theta series

$$\theta_{k-\frac{1}{2}}(z) = (2\pi y)^{-\frac{k-1}{2}} \sum_{\substack{n\geq 1\\ n \text{ odd}}} \left(\frac{-4}{n}\right)^{k-1} H_{k-1}(n\sqrt{\pi y/2})e(n^2 z/8), \quad y = \mathrm{Im}(z) > 0,$$

where

$$H_{k-1}(x) = \sum_{0\leq j\leq (k-1)/2} \frac{(k-1)!}{j!((k-1)-2j)!}(-1)^j(2x)^{(k-1)-2j}$$

is the Hermite polynomial of degree $k-1$ and $e(z) = e^{2\pi i z}$. In Proposition 6.1 we will prove that for each even integer $k \geq 2$ the theta series $\theta_{k-\frac{1}{2}}(z)$ is a real analytic modular form (holomorphic for $k = 2$) of weight $k - \frac{1}{2}$ for $SL_2(\mathbb{Z})$ with multiplier system and exponential decay in the cusp at ∞ of the modular curve $X = SL_2(\mathbb{Z})\backslash\mathbb{H}$. Define the square of the Peterson norm of $\theta_{k-\frac{1}{2}}(z)$ by

$$\langle \theta_{k-\frac{1}{2}}, \theta_{k-\frac{1}{2}} \rangle_{\mathrm{Pet}} = \int_X |\theta_{k-\frac{1}{2}}(z)|^2 \mathrm{Im}(z)^{k-\frac{1}{2}} d\mu(z),$$

where $d\mu(z) = (3/\pi)dxdy/y^2$ is the invariant hyperbolic measure of mass 1 on X.

The main result of this paper is the following asymptotic formula for the average of $L(\psi_k, k)/L(\left(\frac{-D}{\cdot}\right), 1)$ over Ψ_k as $D \to \infty$.

Theorem A. *Let $k \geq 2$ be an even integer and assume that $(-1)^{\frac{D-3}{4}} = 1$ (i.e. the root number of ψ_k is 1). Then for all $\delta < 1/28$,*

$$\frac{1}{h(-D)} \sum_{\psi_k \in \Psi_k} \frac{L(\psi_k, k)}{L(\left(\frac{-D}{\cdot}\right), 1)} = \frac{2^{\frac{5}{2}}\pi^{k-1}}{(k-1)!}\langle \theta_{k-\frac{1}{2}}, \theta_{k-\frac{1}{2}} \rangle_{\mathrm{Pet}} + O_{k,\delta}(D^{-\delta})$$

as $D \to \infty$. The implied constant in the term $O_{k,\delta}(D^{-\delta})$ is ineffective.

We will use Theorem A to prove the following quantitative nonvanishing theorem for central values in the family $\{L(\psi_k, k) : \ \psi_k \in \Psi_k\}$.

Corollary B. *Let $k \geq 2$ be an even integer and assume that $(-1)^{\frac{D-3}{4}} = 1$. Then for all $\delta < 1/60$,*

$$|\{\psi_k \in \Psi_k : \ L(\psi_k, k) > 0\}| \gg_{k,\delta} D^{\delta}$$

as $D \to \infty$. The implied constant in the lower bound $\gg_{k,\delta}$ is ineffective.

Remark 1.1. Results similar to those in Theorem A and Corollary B have been obtained recently by the author and Tonghai Yang [MaY, M] for central values and central derivatives of L–series associated to *quadratic twists* of *subfamilies* of Hecke characters ψ_k in Ψ_k.

Remark 1.2. In [MV], Michel and Venkatesh proved (among other things) that if q is a prime which is inert in K and f is a holomorphic Hecke-eigen cuspform of weight 2 on $\Gamma_0(q)$, then for all $\delta < 1/2700$,

$$\left|\{\xi \in \mathrm{CL}(K)^{\wedge} : \ L(f \otimes \xi, 1/2) \neq 0\}\right| \gg_{\delta,f} D^{\delta}$$

as $D \to \infty$. Here the L–function $L(f \otimes \xi, s)$ is the Rankin-Selberg convolution of f with the theta series $\theta_\xi(z) = \sum_{0 \neq \mathfrak{a} \subset \mathcal{O}_K} \xi(\mathfrak{a}) e(N(\mathfrak{a})z)$.

Remark 1.3. By a theorem of Rogawski [Ro] (see also [Y1]),

$$\frac{L(\psi_k, k)}{L(\left(\frac{-D}{\cdot}\right), 1)} = c\,|\theta_\phi(\eta_{k-1})(1)|^2,$$

where c is an explicit constant, $\phi \in S(\mathbb{Q}_{\mathbb{A}})$ is a Schwartz function on the adele ring $\mathbb{Q}_{\mathbb{A}}$, η_{k-1} is a character (depending on ψ_k) of the norm 1 subgroup $K^1_{\mathbb{A}}$ of the adele ring $K_{\mathbb{A}}$, and $\theta_\phi(\eta_{k-1})(1)$ is an integral over $K^1 \backslash K^1_{\mathbb{A}}$ given by a theta lift from the unitary group $U(1)$ to itself.

The Hecke characters ψ_k are examples of *canonical* Hecke characters of K in the sense of Rohrlich [R2]. Interest in these characters is motivated in part by the fact that the order of vanishing of $L(\psi_k, s)$ at $s = 1$ is related to the Mordell-Weil rank of $\mathbb{Q}$-curves (see e.g. [MY]). More generally, the order of vanishing of $L(\psi_k, s)$ at $s = k$ for $k \geq 2$ is related by the Bloch-Beilinson conjecture to the dimension of the image of an étale version of the Abel-Jacobi map on the Chow group of a Kuga-Sato variety (see section 2).

The nonvanishing of the central values $L(\psi_k, k)$ and their quadratic twists has been studied extensively in [R1, R2, MR, RV1, RV2, RVY, Y2, MY, LX]. In these papers it is assumed that there is *one* orbit of the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$ on Ψ_k. This allows these authors to prove there exists a $\psi_k \in \Psi_k$ for which $L(\psi_k, k) \neq 0$, and to use a well-known theorem of Shimura [Sh] to conclude that $L(\psi_k, k) \neq 0$ for *all* $\psi_k \in \Psi_k$. The strongest possible result under this hypothesis is due to Yang [Y2].

If there is *more than one* orbit of the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$ on Ψ_k (which happens if $\gcd(2k-1, h(-D)) > 1$), the existence of one nonvanishing central value no longer implies that all of the central values are nonvanishing. It then becomes of interest to understand how the number of nonvanishing central values in the family $\{L(\psi_k, k) : \psi_k \in \Psi_k\}$ grows with the discriminant $-D$. Corollary B is the first result of this type.

Note that if $k = 2$ and $D = 59$ so that $h(-59) = 3$ and the root number $w(\psi_k) = (-1)^{16} = 1$, Rodriguez-Villegas [RV2] found that there are two distinct L–series in $\{L(\psi_2, s) : \psi_2 \in \Psi_2\}$ which *vanish* at $s = 2$ to order at least 2 (the corresponding cusp forms f_{ψ_2} have weight 4). One reason for interest in examples of this type is that primitive cusp forms whose L–series have high order of vanishing at the critical center $s = 1/2$ yield effective lower bounds for the class number $h(-D)$ (see [IK, pg. 541]).

To prove Theorem A we use a combination of arithmetic and spectral methods. Essential use is made of the following two deep results: a formula of Rodriguez-Villegas and Zagier [RV1, RVZ, RV2] for the central value $L(\psi_k, k)$, and Duke's theorem [D] on the equidistribution of CM points with respect to the measure $d\mu(z)$ on X. The Rodriguez-Villegas-Zagier formula expresses $L(\psi_k, k)$ as the absolute value squared of a linear combination of non-holomorphic derivatives of half-integral weight theta series evaluated at CM points (see Theorem 4.1). One immediate consequence is that $L(\psi_k, k) \geq 0$.

To prove Corollary B we combine Theorem A with a deep subconvexity bound of Duke, Friedlander, and Iwaniec [DFI1] for $GL(2)$ automorphic L–functions associated to normalized cusp newforms f of weight $k \geq 2$ and level q with trivial nebentypus.

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2. The Bloch-Beilinson conjecture

One reason for interest in the central values of L–series of Hecke characters of weight > 1 is that there exists a natural generalization of the Birch and Swinnerton-Dyer conjecture to L–functions of motives of CM forms.

Let $\psi_{k,u}$ be the unitary Hecke character of K of conductor $\sqrt{-D}\mathcal{O}_K$ associated to ψ_k (see section 8). By Hecke's converse theorem, $\psi_{k,u}$ corresponds to a normalized, primitive, CM newform $f_{\psi_{k,u}}$ of weight $2k$ and level D^2. Let $F = \mathbb{Q}(a(1), a(2), \ldots)$ be the totally real field generated by the Fourier coefficients of $f_{\psi_{k,u}}$. By Scholl [S] and Nekovar [N, Proposition 3.1] one can associate to $f_{\psi_{k,u}}$ a motive $M = M(f_{\psi_{k,u}})$ over $\mathbb{Q}$ with coefficients in F and an L-function $L(M(-1), s)$ such that the L–series

$$L(M(-1), s) = L(f_{\psi_{k,u}}, s + k - 1) = \sum_{n=1}^{\infty} a(n) n^{-s-k+1}.$$

Let $N \geq 3$ be an integer. Let $Y = \bar{\bar{X}}_N^{2k-2}$ be the nonsingular compactification of the $(2k-1)$-dimensional Kuga-Sato variety over the modular curve M_N over $\mathbb{Q}$ parameterizing elliptic curves with full level N structure. Let p be a prime not dividing $2(2k-2)!N\phi(N)$. One can define an étale version of the Abel-Jacobi map (see [N, pg. 104])

$$\Phi : CH^k(Y/K)_0 \longrightarrow H^1_{\text{cont}}(K, H^{2k-1}_{\text{et}}(Y \otimes \overline{K}, \mathbb{Z}_p(k)))$$

where $CH^k(Y/K)_0$ is the Chow group of homologically trivial cycles of codimension k on Y defined over K, modulo rational equivalence. The Abel-Jacobi map Φ induces a map

$$\Phi_{f_{\psi_{k,u}},K} : \prod (CH^k(Y/K)_0 \otimes \mathbb{Z}_p) \longrightarrow H^1_{\text{cont}}(K, A)$$

where $\prod$ is a certain projector and A is a Galois module which is a higher weight analogue of the Tate module of a modular elliptic curve (see [N, pg. 105]).

One version of the Bloch-Beilinson conjecture relates the order of vanishing of $L(M(-1), s)$ at the critical center $s = 1$ to the dimension of the image of the induced map $\Phi_{f_{\psi_{k,u}},K}$.

Conjecture 2.1 (Bloch-Beilinson, cf. [J])**.**

$$\dim_{\mathbb{Q}_p}(\text{Im}(\Phi_{f_{\psi_{k,u}},K}) \otimes \mathbb{Q}) = \text{ord}_{s=k} L(f_{\psi_{k,u}}, s) \ (= \text{ord}_{s=k} L(\psi_k, s)).$$

3. Genus theory

In this section we elaborate on [RV2, section 3], where a genus $\mathfrak{G}_{\psi_k}$ is associated to each Hecke character $\psi_k \in \Psi_k$. In particular, we will use the map (3.1) to perform the averaging procedure in section 5.

Each element of $\text{CL}_2(K)$ is associated to a factorization of the discriminant $-D$ as follows. Given $U \in \text{CL}_2(K)$, there are ideals $\mathfrak{D}_1$ and $\mathfrak{D}_2$ such that $\sqrt{-D}\mathcal{O}_K = \mathfrak{D}_1 \cdot \mathfrak{D}_2$ and $U = [\mathfrak{D}_1] = [\mathfrak{D}_2]$. Let $D_1 = N(\mathfrak{D}_1)$ and $D_2 = N(\mathfrak{D}_2)$, so that $-D = D_1 \cdot (-D_2)$. Choose $D_1 \equiv 1 \bmod 4$ and $D_2 \equiv 3 \bmod 4$, determining them uniquely. Note that $D_1 > 0$, $D_2 > 0$, and $\gcd(D_1, D_2) = 1$.

One can associate to each such factorization a genus character by defining

$$\chi_U([\mathfrak{a}]) = \left(\frac{N(\mathfrak{a})}{D_1}\right), \quad \mathfrak{a} \in [\mathfrak{a}] \text{ prime to } \sqrt{-D}\mathcal{O}_K.$$

This map is a homomorphism in each variable. A genus is a set of ideals of K whose classes lie in a fixed coset of $\text{CL}^{(2)}(K)$. The genus group is $\text{CL}(K)/\text{CL}^2(K)$. Every character $\phi \in \text{CL}_2(K)^\wedge$ determines a unique genus $\mathfrak{G}_\phi$ as follows. Define a map

$$\text{CL}_2(K)^\wedge = \{\phi : \text{CL}_2(K) \to \{\pm 1\}\} \longrightarrow \text{CL}(K)/\text{CL}^2(K)$$

by

$$\phi \to \mathfrak{G}_\phi = \{\mathfrak{a} : \ \chi_U([\mathfrak{a}]) \cdot \phi(U) = 1, \text{ for every } U \in \text{CL}_2(K)\}. \tag{3.1}$$

One can use genus theory to show that the map $\phi \to \mathfrak{G}_\phi$ is a bijection.

Proposition 3.1. *The map defined by* (3.1) *is a bijection.*

Proof. First, we fix some notation. For a number field L and ideal $\mathfrak{a} \subset \mathcal{O}_L$, let

$$Id_L(\mathfrak{a}) = \{\text{fractional ideals of } L \text{ prime to } \mathfrak{a}\}.$$

Let H be the Hilbert class field of K.

By genus theory, there exists a bijection

$$\{D_1 : \ D_1 | D, \ D_1 \equiv 1 \bmod 4\} \longrightarrow \{\text{unramified quadratic extensions of } K = \mathbb{Q}(\sqrt{-D})\}$$

given by

$$D_1 \longmapsto K(\sqrt{D_1}).$$

We have the following diagram:

$$\begin{array}{ccc}
\mathrm{CL}(K) \cong \mathrm{Gal}(H/K) & \xrightarrow{\mathrm{Res}} & \mathrm{Gal}(K(\sqrt{D_1})/K) \\
\Big\downarrow{\scriptstyle N_{K/\mathbb{Q}}(\cdot)} & & \Big\downarrow{\scriptstyle \mathrm{Res}} \\
Id_{\mathbb{Q}}(D_1) & \xrightarrow{\mathrm{Artin}} & \mathrm{Gal}(\mathbb{Q}(\sqrt{D_1})/\mathbb{Q}) \\
& \searrow{\scriptstyle \left(\frac{\cdot}{D_1}\right)} & \Big\downarrow{\scriptstyle \mathrm{can}} \\
& & \{\pm 1\}
\end{array}$$

Again, by genus theory, $D_1\mathcal{O}_K = \mathfrak{D}_1^2$ for some $\mathfrak{D}_1 \in Id_K(D)$ and $\{\mathfrak{D}_1\} = \mathrm{CL}_2(K)$. We have a pairing

$$Id_K(D) \times \mathrm{CL}_2(K) \longrightarrow \{\pm 1\}$$

given by

$$\mathfrak{a} \times \mathfrak{D}_1 \longmapsto \left(\frac{N(\mathfrak{a})}{N(\mathfrak{D}_1)}\right) = \left(\frac{N(\mathfrak{a})}{D_1}\right).$$

By the preceding facts, this pairing factorises to a perfect pairing

$$\mathrm{CL}(K)/\mathrm{CL}^2(K) \times \mathrm{CL}_2(K) \longrightarrow \{\pm 1\},$$

i.e. the induced map

$$\mathrm{CL}_2(K) \longrightarrow \big(\mathrm{CL}(K)/\mathrm{CL}^2(K)\big)^{\wedge}$$

given by

$$\mathfrak{D}_1 \longmapsto \left(\frac{N(\cdot)}{D_1}\right)$$

is a bijection. □

Choose a primitive ideal $\mathfrak{a}$ prime to $\sqrt{-D}\mathcal{O}_K$ such that $\mathfrak{a}\mathfrak{D}_1 = (\mu)$, where $\mu = (mD_1 + n\sqrt{-D})/2$ with integers m, n of the same parity. Given a Hecke character $\psi_k \in \Psi_k$, define

$$\chi_{\psi_k}(U) = \left(\frac{2m}{D_2}\right)\left(\frac{n}{D_1}\right)\frac{(\mu/\sqrt{D_1})}{\psi_k(\mathfrak{a})}, \tag{3.2}$$

where $\sqrt{D_1} > 0$. Because $4N(\mathfrak{a}) = m^2D_1 + N^2D_2$ is prime to D, m is prime to D_2 and n is prime to D_1, and therefore $\chi_{\psi_k}(U) \neq 0$.

By [RV2, Proposition A], the map

$$\chi_{\psi_k} : \mathrm{CL}_2(K) \longrightarrow \{\pm 1\}$$

defined by (3.2) is a well-defined homomorphism. Therefore, by (3.1) one can associate to each ψ_k a unique genus

$$\mathfrak{G}_{\psi_k} = \{\mathfrak{a} : \ \chi_U([\mathfrak{a}]) \cdot \chi_{\psi_k}(U) = 1, \text{ for every } U \in \mathrm{CL}_2(K)\}.$$

Note that for any character ξ of $\mathrm{CL}(K)$, $\chi_{\xi\psi_k} = \xi\chi_{\psi_k}$, where ξ here is restricted to $\mathrm{CL}_2(K)$.

4. A Formula for the central value $L(\psi_k, k)$

In this section we state a formula of Rodriguez-Villegas [RV2] for the central value $L(\psi_k, k)$.

In order to evaluate the theta series $\theta_{k-\frac{1}{2}}(z)$ on CM points, Rodriguez-Villegas and Zagier impose congruence conditions on the bases of ideals (see [RVZ, pgs. 86–87 and pg. 89]). Fix the choice

$$z_{\mathfrak{a}}^{(2)} = \frac{b+\sqrt{-D}}{2a} \in \mathbb{H}$$

with

$$\mathfrak{a} = \left[a, \frac{b+\sqrt{-D}}{2}\right], \quad (a,2)=1, \quad \text{and} \quad b \equiv 1 \bmod 16,$$

which is well-defined modulo $8\mathbb{Z}$.

Define $\delta = 0$ or 1 by $(D+1)/4 \equiv \delta \bmod 2$, or equivalently, $(-1)^\delta = (-D/2)$.

Theorem 4.1 (Rodriguez-Villegas [RV2])**.** *Let k be a positive integer, $\psi_k \in \Psi_k$, $\mathfrak{G}_{\psi_k}$ be the genus associated to ψ_k, $\mathfrak{a}_1$ be a primitive ideal in $\mathfrak{G}_{\psi_k}$ prime to $\sqrt{-D}\mathcal{O}_K$, and $t(D)$ be the number of prime factors of D. Then*

$$L(\psi_k, k) = c_k \frac{2^{t(D)-1}}{N(\mathfrak{a}_1)^{k-\frac{1}{2}}} \left| \sum_{[\mathfrak{a}]\in \mathrm{CL}^{(2)}(K)} \left(\frac{-4}{N(\mathfrak{a})}\right)^{\delta} \overline{\psi_k(\mathfrak{a})}^{-1} \theta_{k-\frac{1}{2}}(z_{\mathfrak{a}_1\mathfrak{a}^2}^{(2)}) \right|^2, \tag{4.1}$$

where

$$c_k = \frac{\pi^k D^{\frac{k}{2}-\frac{3}{4}}}{2^{k-3}(k-1)!}.$$

Remark 4.2. In Remark (1) on [RV2, pg. 436] it is explained that

$$\left(\frac{-4}{N(\mathfrak{a})}\right)^{\delta} \overline{\psi_k(\mathfrak{a})}^{-1} \theta_{k-\frac{1}{2}}(z_{\mathfrak{a}_1\mathfrak{a}^2}^{(2)})$$

depends only on ψ_k, $\mathfrak{a}_1$, and the ideal class of $\mathfrak{a}$, and that the right hand side of (4.1) is independent of $\mathfrak{a}_1$. See also the corollary on [RVZ, pg. 90].

5. The average value of $L(\psi_k, k)$

In this section we use Theorem 4.1 to obtain an explicit formula for the average of $L(\psi_k, k)$ over Ψ_k.

Proposition 5.1. *For each integer $k \geq 1$ we have*

$$\frac{1}{h(-D)} \sum_{\psi_k \in \Psi_k} L(\psi_k, k) = c_k \sum_{[\mathfrak{a}]\in \mathrm{CL}(K)} \frac{|\theta_{k-\frac{1}{2}}(z_{\mathfrak{a}}^{(2)})|^2}{N(\mathfrak{a})^{k-\frac{1}{2}}}. \tag{5.1}$$

Proof. Fix a Hecke character $\psi_{k,0} \in \Psi_k$. Then $\Psi_k = \{\psi_{k,0}\xi : \ \xi \in \mathrm{CL}(K)^\wedge\}$, which yields the decomposition

$$\sum_{\psi_k\in\Psi_k} L(\psi_k, k) = \sum_{\xi \in \mathrm{CL}(K)^\wedge} L(\psi_{k,0}\xi, k) = \sum_{\phi\in \mathrm{CL}_2(K)^\wedge} \sum_{\substack{\xi\in \mathrm{CL}(K)^\wedge \\ \xi|_{\mathrm{CL}_2(K)}=\phi}} L(\psi_{k,0}\xi, k).$$

Recall that $\mathfrak{a}_1 \in \mathfrak{G}_{\psi_k}$ depends on ψ_k (and hence on ϕ). We now expand (4.1) and use

$$\overline{\psi_{k,0}\xi(\mathfrak{a})} = \psi_{k,0}\xi(\bar{\mathfrak{a}})$$

and

$$\psi_{k,0}(\bar{\mathfrak{a}})\psi_{k,0}(\mathfrak{a}) = N(\mathfrak{a})^{2k-1}$$

to obtain

$$\begin{aligned}
&\sum_{\substack{\xi\in\mathrm{CL}(K)^\wedge \\ \xi|_{\mathrm{CL}_2(K)}=\phi}} L(\psi_{k,0}\xi, k) \\
&= \frac{c_k 2^{t(D)-1}}{N(\mathfrak{a}_1(\phi))^{k-\frac{1}{2}}} \sum_{\substack{\xi\in\mathrm{CL}(K)^\wedge \\ \xi|_{\mathrm{CL}_2(K)}=\phi}} \sum_{[\mathfrak{a}],[\mathfrak{a}']\in\mathrm{CL}^{(2)}(K)} \left(\frac{-4}{N(\mathfrak{a})}\right)^\delta \left(\frac{-4}{N(\mathfrak{a}')}\right)^\delta \frac{\theta_{k-\frac{1}{2}}(z^{(2)}_{\mathfrak{a}_1(\phi)\mathfrak{a}^2})\overline{\theta_{k-\frac{1}{2}}(z^{(2)}_{\mathfrak{a}_1(\phi)\mathfrak{a}'^2})}}{\overline{\psi_{k,0}\xi(\mathfrak{a})}\psi_{k,0}\xi(\mathfrak{a}')} \\
&= \frac{c_k 2^{t(D)-1}}{N(\mathfrak{a}_1(\phi))^{k-\frac{1}{2}}} \left\{ \sum_{[\mathfrak{a}]\in\mathrm{CL}^{(2)}(K)} \frac{\left|\theta_{k-\frac{1}{2}}(z^{(2)}_{\mathfrak{a}_1(\phi)\mathfrak{a}^2})\right|^2}{\psi_{k,0}(\bar{\mathfrak{a}})\psi_{k,0}(\mathfrak{a})} \sum_{\substack{\xi\in\mathrm{CL}(K)^\wedge \\ \xi|_{\mathrm{CL}_2(K)}=\phi}} \frac{1}{\xi(\bar{\mathfrak{a}})\xi(\mathfrak{a})} \right. \\
&\qquad \left. + \sum_{\substack{[\mathfrak{a}],[\mathfrak{a}']\in\mathrm{CL}^{(2)}(K) \\ [\mathfrak{a}]\neq[\mathfrak{a}']}} \left(\frac{-4}{N(\mathfrak{a}\mathfrak{a}')}\right)^\delta \frac{\theta_{k-\frac{1}{2}}(z^{(2)}_{\mathfrak{a}_1(\phi)\mathfrak{a}^2})\overline{\theta_{k-\frac{1}{2}}(z^{(2)}_{\mathfrak{a}_1(\phi)\mathfrak{a}'^2})}}{\psi_{k,0}(\bar{\mathfrak{a}})\psi_{k,0}(\mathfrak{a}')} \sum_{\substack{\xi\in\mathrm{CL}(K)^\wedge \\ \xi|_{\mathrm{CL}_2(K)}=\phi}} \frac{1}{\xi(\bar{\mathfrak{a}})\xi(\mathfrak{a}')} \right\} \\
&= \frac{c_k 2^{t(D)-1}}{N(\mathfrak{a}_1(\phi))^{k-\frac{1}{2}}} \left\{ \sum_{[\mathfrak{a}]\in\mathrm{CL}^{(2)}(K)} \frac{\left|\theta_{k-\frac{1}{2}}(z^{(2)}_{\mathfrak{a}_1(\phi)\mathfrak{a}^2})\right|^2}{N(\mathfrak{a})^{2k-1}} S(\phi, \mathfrak{a}, \mathfrak{a}) \right. \\
&\qquad \left. + \sum_{\substack{[\mathfrak{a}],[\mathfrak{a}']\in\mathrm{CL}^{(2)}(K) \\ [\mathfrak{a}]\neq[\mathfrak{a}']}} \left(\frac{-4}{N(\mathfrak{a}\mathfrak{a}')}\right)^\delta \frac{\theta_{k-\frac{1}{2}}(z^{(2)}_{\mathfrak{a}_1(\phi)\mathfrak{a}^2})\overline{\theta_{k-\frac{1}{2}}(z^{(2)}_{\mathfrak{a}_1(\phi)\mathfrak{a}'^2})}}{\psi_{k,0}(\bar{\mathfrak{a}})\psi_{k,0}(\mathfrak{a}')} S(\phi, \mathfrak{a}, \mathfrak{a}') \right\},
\end{aligned}$$

where

$$S(\phi, \mathfrak{a}, \mathfrak{a}') := \sum_{\substack{\xi\in\mathrm{CL}(K)^\wedge \\ \xi|_{\mathrm{CL}_2(K)}=\phi}} \frac{1}{\xi(\bar{\mathfrak{a}})\xi(\mathfrak{a}')}.$$

By orthogonality,

$$S(\phi, \mathfrak{a}, \mathfrak{a}') = |\mathrm{CL}^{(2)}(K)|\delta_{\mathfrak{a},\mathfrak{a}'}$$

where

$$\delta_{\mathfrak{a},\mathfrak{a}'} = \begin{cases} 1, & [\mathfrak{a}] = [\mathfrak{a}'] \\ 0, & [\mathfrak{a}] \neq [\mathfrak{a}'], \end{cases}$$

and by genus theory (see e.g. [I2, pg. 221]),

$$2^{t(D)-1}|\mathrm{CL}^{(2)}(K)| = |\mathrm{CL}_2(K)| \cdot |\mathrm{CL}^{(2)}(K)| = h(-D).$$

Hence the inner sum simplifies to

$$\sum_{\substack{\xi \in \mathrm{CL}(K)^\wedge \\ \xi_{|\mathrm{CL}_2(K)} = \phi}} L(\psi_{k,0}\xi, k) = \frac{c_k h(-D)}{N(\mathfrak{a}_1(\phi))^{k-\frac{1}{2}}} \sum_{[\mathfrak{a}] \in \mathrm{CL}^{(2)}(K)} \frac{\left|\theta_{k-\frac{1}{2}}(z^{(2)}_{\mathfrak{a}_1(\phi)\mathfrak{a}^2})\right|^2}{N(\mathfrak{a})^{2k-1}},$$

or equivalently,

$$\sum_{\substack{\xi \in \mathrm{CL}(K)^\wedge \\ \xi_{|\mathrm{CL}_2(K)} = \phi}} L(\psi_{k,0}\xi, k) = c_k h(-D) \sum_{[\mathfrak{a}] \in \mathrm{CL}^{(2)}(K)} N(\mathfrak{a}_1(\phi)\mathfrak{a}^2)^{\frac{1}{2}-k} \left|\theta_{k-\frac{1}{2}}(z^{(2)}_{\mathfrak{a}_1(\phi)\mathfrak{a}^2})\right|^2.$$

Using that the map $\phi \to \mathfrak{G}_\phi$ defined by (3.1) is a bijection, we sum over all $\phi \in \mathrm{CL}_2(K)^\wedge$ in the inner sum to obtain

$$\begin{aligned}
&\sum_{\phi \in \mathrm{CL}_2(K)^\wedge} \sum_{\substack{\xi \in \mathrm{CL}(K)^\wedge \\ \xi_{|\mathrm{CL}_2(K)} = \phi}} L(\psi_{k,0}\xi, k) \\
&= c_k h(-D) \sum_{\phi \in \mathrm{CL}_2(K)^\wedge} \sum_{[\mathfrak{a}] \in \mathrm{CL}^{(2)}(K)} N(\mathfrak{a}_1(\phi)\mathfrak{a}^2)^{\frac{1}{2}-k} \left|\theta_{k-\frac{1}{2}}(z^{(2)}_{\mathfrak{a}_1(\phi)\mathfrak{a}^2})\right|^2 \\
&= c_k h(-D) \sum_{[\mathfrak{a}_1] \in \mathrm{CL}(K)/\mathrm{CL}^2(K)} \sum_{[\mathfrak{a}] \in \mathrm{CL}^2(K)} N(\mathfrak{a}_1\mathfrak{a}^2)^{\frac{1}{2}-k} \left|\theta_{k-\frac{1}{2}}(z^{(2)}_{\mathfrak{a}_1\mathfrak{a}^2})\right|^2 \\
&= c_k h(-D) \sum_{[\mathfrak{a}] \in \mathrm{CL}(K)} \frac{|\theta_{k-\frac{1}{2}}(z^{(2)}_{\mathfrak{a}})|^2}{N(\mathfrak{a})^{k-\frac{1}{2}}}.
\end{aligned}$$

□

6. Derivatives of modular forms

In this section we review some facts from [RVZ, Section 2] and then prove that for each even integer $k \geq 2$ the theta series $\theta_{k-\frac{1}{2}}(z)$ is a weight $k - \frac{1}{2}$ modular form for $SL_2(\mathbb{Z})$ with multiplier system and exponential decay in the cusp at ∞ of $SL_2(\mathbb{Z})\backslash\mathbb{H}$.

Define

$$z^{\frac{1}{2}} = |z|^{\frac{1}{2}} e^{\frac{i\theta}{2}}, \quad \text{if} \quad z = |z|e^{i\theta}, \quad -\pi < \theta \leq \pi,$$

and $z^k = (z^{\frac{1}{2}})^{2k}$ for a half integer $k \in \frac{1}{2} + \mathbb{Z}$. Let $\Gamma \subset SL_2(\mathbb{R})$ be a modular group and let $M_k^*(\Gamma, m)$ be the space of real analytic modular forms $f : \mathbb{H} \to \mathbb{C}$ of weight $k \in \frac{1}{2} + \mathbb{Z}$ for Γ with multiplier system $m : \Gamma \to \mathbb{C}$. Thus, f is a real analytic function satisfying

$$f_{|_k} A = m(A) f \quad \text{for all} \quad A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma,$$

where the slash operator $|_k A$ is defined by

$$f_{|_k} A(z) = (\gamma z + \delta)^{-k} f(Az).$$

Define the differential operators

$$D = \frac{1}{2\pi i}\frac{d}{dz} \tag{6.1}$$

and

$$\partial_k = D - \frac{k}{4\pi y}.$$

It is known that the operator ∂_k satisfies

$$\partial_k(f_{|_k}A) = (\partial_k f)_{|_{k+2}}A \quad \text{for all} \quad A \in \Gamma. \tag{6.2}$$

A simple calculation using (6.2) shows that

$$\partial_k : M_k^*(\Gamma, m) \longrightarrow M_{k+2}^*(\Gamma, m).$$

More generally, one can define the differential operators

$$\partial_k^\ell = \partial_{k+2\ell-2} \circ \partial_{k+2\ell-4} \circ \cdots \circ \partial_{k+2} \circ \partial_k, \quad \ell \in \mathbb{Z}_{\geq 0},$$

and then

$$\partial_k^\ell : M_k^*(\Gamma, m) \longrightarrow M_{k+2\ell}^*(\Gamma, m). \tag{6.3}$$

The following formula for ∂_k^ℓ can be proved by induction (see [B]),

$$\partial_k^\ell = \sum_{j=0}^{\ell} \binom{\ell}{j} \frac{\Gamma(\ell+k)}{\Gamma(j+k)} \left(\frac{-1}{4\pi y}\right)^{\ell-j} D^j. \tag{6.4}$$

In particular, if $f \in M_k^*(\Gamma, m)$ has a Fourier expansion of the form

$$\sum_{n=0}^{\infty} a(n)e(nz),$$

then

$$\partial_k^\ell \left(\sum_{n=0}^{\infty} a(n)e(nz)\right) = \frac{(-1)^\ell \ell!}{(4\pi y)^\ell} \sum_{n=0}^{\infty} a(n) L_\ell^{k-1}(4\pi n y)e(nz) \tag{6.5}$$

where

$$L_\ell^\alpha(z) = \sum_{j=0}^{\ell} \binom{\ell+\alpha}{\ell-j} \frac{(-z)^j}{j!}, \quad \alpha \in \mathbb{C},$$

is the ℓ-th generalized Laguerre polynomial. Note that for $k = 1/2$ one has the identity

$$L_\ell^{-\frac{1}{2}}(z) = (-1/4)^\ell H_{2\ell}(\sqrt{z})/\ell!, \tag{6.6}$$

and for $k = 3/2$ one has the identity

$$L_\ell^{\frac{1}{2}}(z) = (-1/4)^\ell H_{2\ell+1}(\sqrt{z})/2\sqrt{z}\ell!, \tag{6.7}$$

where

$$H_p(x) = \sum_{0 \leq j \leq p/2} \frac{p!}{j!(p-2j)!}(-1)^j (2x)^{p-2j}, \quad p \in \mathbb{Z}_{\geq 0},$$

is the p-th Hermite polynomial.

Define the weight 1/2 holomorphic theta series

$$\theta_{1/2}(z) = \sum_{\substack{n\geq 1 \\ n \text{ odd}}} e(n^2 z/8),$$

and the weight 3/2 holomorphic theta series

$$\theta_{3/2}(z) = \sum_{\substack{n\geq 1 \\ n \text{ odd}}} \left(\frac{-4}{n}\right) n e(n^2 z/8).$$

Using (6.5) and (6.6) one obtains

$$\begin{aligned}
\partial_{1/2}^{\ell}\theta_{1/2}(z) &= \frac{(-1)^{\ell}\ell!}{(4\pi y)^{\ell}} \sum_{\substack{n\geq 1 \\ n \text{ odd}}} L_{\ell}^{-1/2}(\pi n^2 y/2) e(n^2 z/8) \\
&= \frac{1}{(16\pi y)^{\ell}} \sum_{\substack{n\geq 1 \\ n \text{ odd}}} H_{2\ell}(n\sqrt{\pi y/2}) e(n^2 z/8).
\end{aligned}$$

Similarly, using (6.5) and (6.7) one obtains

$$\begin{aligned}
\partial_{3/2}^{\ell}\theta_{3/2}(z) &= \frac{(-1)^{\ell}\ell!}{(4\pi y)^{\ell}} \sum_{\substack{n\geq 1 \\ n \text{ odd}}} \left(\frac{-4}{n}\right) n L_{\ell}^{1/2}(\pi n^2 y/2) e(n^2 z/8) \\
&= \frac{1}{\sqrt{2\pi y}} \frac{1}{(16\pi y)^{\ell}} \sum_{\substack{n\geq 1 \\ n \text{ odd}}} \left(\frac{-4}{n}\right) H_{2\ell+1}(n\sqrt{\pi y/2}) e(n^2 z/8).
\end{aligned}$$

These computations show that the Fourier expansions of the functions $\theta_{p+\frac{1}{2}}(z)$ defined by

$$\theta_{p+\frac{1}{2}}(z) = \begin{cases} 8^{\ell}\partial_{1/2}^{\ell}\theta_{1/2}(z) & \text{if } p = 2\ell,\ \ell \in \mathbb{Z}_{\geq 0}, \\ 8^{\ell}\partial_{3/2}^{\ell}\theta_{3/2}(z) & \text{if } p = 2\ell+1,\ \ell \in \mathbb{Z}_{\geq 0}, \end{cases}$$

can be given by the uniform formula

$$\theta_{p+\frac{1}{2}}(z) = \frac{1}{(2\pi y)^{p/2}} \sum_{\substack{n\geq 1 \\ n \text{ odd}}} \left(\frac{-4}{n}\right)^{p} H_p(n\sqrt{\pi y/2}) e(n^2 z/8).$$

Let $S_k^*(\Gamma, m) \subset M_k^*(\Gamma, m)$ be the subset of modular forms with exponential decay in each cusp of $\Gamma\backslash\mathbb{H}$.

Proposition 6.1. (*i*) *The theta series*

$$\theta_{\frac{1}{2}+2\ell} \in M^*_{\frac{1}{2}+2\ell}(\Gamma_0(2), \lambda) \quad \textit{for} \quad \ell \in \mathbb{Z}_{\geq 0},$$

where the multiplier system λ *is given by*

$$\lambda(A) = \left(\frac{\gamma}{\delta}\right) e_8(\rho(A)) \quad \textit{if} \quad A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_0(2),$$

where $\left(\frac{\gamma}{\delta}\right)$ *is the Jacobi symbol if* $0 < \delta \equiv 1 \bmod 2$, *extended to all* $\delta \equiv 1 \bmod 2$ *by*

$$\left(\frac{\gamma}{\delta}\right) = \frac{\gamma}{|\gamma|}\left(\frac{\gamma}{-\delta}\right) \quad \text{if} \quad \gamma \neq 0,$$

$$\left(\frac{0}{\delta}\right) = \begin{cases} 1 & \delta = \pm 1 \\ 0 & \text{otherwise,} \end{cases}$$

$\rho(A) = \delta - 1 + \delta\beta$, *and* $e_8(n) = e(n/8)$ *for* $n \in \mathbb{Z}$.

(*ii*) *The theta series*

$$\theta_{\frac{3}{2}+2\ell} \in S^*_{\frac{3}{2}+2\ell}(SL_2(\mathbb{Z}), \vartheta^3) \quad \text{for} \quad \ell \in \mathbb{Z}_{\geq 0},$$

where ϑ *is the multiplier system for the Dedekind eta function* $\eta(z)$ *(see equation (6.8)).*

Proof. (*i*) It is known that the Jacobi theta series $\theta_{1/2}(z)$ is a holomorphic modular form of weight $1/2$ for $\Gamma_0(2)$ with multiplier system λ (see [W]). Thus we have $\theta_{1/2} \in M^*_{1/2}(\Gamma_0(2), \lambda)$, and by (6.3)

$$\partial^\ell_{1/2} : M^*_{1/2}(\Gamma_0(2), \lambda) \longrightarrow M^*_{\frac{1}{2}+2\ell}(\Gamma_0(2), \lambda),$$

so that

$$\theta_{\frac{1}{2}+2\ell}(z) = 8^\ell \partial^\ell_{1/2}\theta_{1/2}(z) \in M^*_{\frac{1}{2}+2\ell}(\Gamma_0(2), \lambda)$$

for all $\ell \in \mathbb{Z}_{\geq 0}$.

(*ii*) First observe that the theta series $\theta_{3/2}(z)$ can be expressed as

$$\theta_{3/2}(z) = \eta^3(z),$$

where

$$\eta(z) = e(z/24)\prod_{n=1}^{\infty}(1 - e(nz))$$

is the Dedekind eta function. It is known that $\eta(z)$ is a holomorphic cusp form of weight $1/2$ for $SL_2(\mathbb{Z})$ with multiplier system ϑ (see [I2, pg. 45]). In particular,

$$\eta(Az) = \vartheta(A)(\gamma z + \delta)^{\frac{1}{2}}\eta(z) \quad \text{if} \quad A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL_2(\mathbb{Z}), \tag{6.8}$$

where $\vartheta(-A) = e(1/4)\vartheta(A)$ for any A, $\vartheta(A) = e(\beta/24)$ if $A = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}$, and

$$\vartheta(A) = e\left(\frac{\alpha + \delta - 3\gamma}{24\gamma} - \frac{1}{2}s(\delta, \gamma)\right) \quad \text{if} \quad A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad \gamma > 0.$$

Here, $s(\delta, \gamma)$ is the Dedekind sum

$$s(\delta, \gamma) = \sum_{0 \leq n < \gamma} \frac{n}{\gamma} P\left(\frac{\delta n}{\gamma}\right),$$

where $P(x) = x - [x] - \frac{1}{2}$.

Because ϑ is a multiplier system of weight $1/2$ for $SL_2(\mathbb{Z})$, ϑ^3 is a multiplier system of weight $3/2$ for $SL_2(\mathbb{Z})$ (see [I2, pg. 42]). It follows from (6.8) that $\theta_{3/2}(z) = \eta^3(z)$ is a holomorphic modular form of weight 3/2 for $SL_2(\mathbb{Z})$ with multiplier system ϑ^3. Thus, we have $\theta_{3/2} \in M^*_{3/2}(SL_2(\mathbb{Z}), \vartheta^3)$, and by (6.3)

$$\partial^\ell_{3/2} : M^*_{3/2}(SL_2(\mathbb{Z}), \vartheta^3) \longrightarrow M^*_{\frac{3}{2}+2\ell}(SL_2(\mathbb{Z}), \vartheta^3),$$

so that

$$\theta_{\frac{3}{2}+2\ell}(z) = 8^\ell \partial^\ell_{3/2}\theta_{3/2}(z) \in M^*_{\frac{3}{2}+2\ell}(SL_2(\mathbb{Z}), \vartheta^3)$$

for all $\ell \in \mathbb{Z}_{\geq 0}$.

It remains to show that $\theta_{\frac{3}{2}+2\ell}(z)$ has exponential decay in the cusp at ∞ of $SL_2(\mathbb{Z})\backslash\mathbb{H}$. Let $f \in M^*_k(\Gamma, m)$ be a holomorphic modular form of weight k for a modular group $\Gamma \subset SL_2(\mathbb{R})$ with multiplier system m (by holomorphic we mean f is holomorphic in $\mathbb{H} \cup \{\text{cusps for } \Gamma\}$). Let x be a cusp for Γ, $\Gamma_x < \Gamma$ be the stabilizer of x, and $\sigma_x \in SL_2(\mathbb{R})$ be a scaling matrix so that $\sigma_x\infty = x$ and $\sigma_x^{-1}\Gamma_x\sigma_x$ is the group of integral translations generated by

$$B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

together with $-B$ if $-I \in \Gamma$. Then $S_x = \sigma_x B \sigma_x^{-1}$ generates the stabilizer Γ_x, together with $-S_x$ if $-I \in \Gamma$.

Write the slash operator $|_k A$ as $|_A$. By the calculation on [I2, pg. 43] we have

$$f_{|\sigma_x}(Bz) = \vartheta(S_x) f_{|\sigma_x}(z). \tag{6.9}$$

Set

$$m(S_x) = e(\kappa_x) \quad \text{with} \quad 0 \leq \kappa_x < 1.$$

Then by (6.9), $e(-\kappa_x z) f_{|\sigma_x}(z)$ is periodic of period 1, and therefore

$$e(-\kappa_x z) f_{|\sigma_x}(z) = g(e(z))$$

where $g(q)$ is holomorphic in the punctured plane $\mathbb{C}^*$. By definition, $f(z)$ is holomorphic at the cusp x if $g(q)$ is holomorphic at $q = 0$. In this case the power series expansion of $g(q)$ at $q = 0$ yields

$$f_{|\sigma_x}(z) = e(\kappa_x z) \sum_{n=0}^{\infty} \widehat{f}_x(n) e(nz), \tag{6.10}$$

where the complex numbers $\widehat{f}_x(n)$ are the Fourier coefficients of $f(z)$ at the cusp x, and

$$\sum_{n=0}^{\infty} \widehat{f}_x(n) e(nz)$$

is the Fourier expansion of $f(z)$ at x.

We now specialize to $f(z) = \theta_{3/2}(z)$, $\Gamma = SL_2(\mathbb{Z})$, and $m = \vartheta^3$ the multiplier system for $\theta_{3/2}(z)$. There is one cusp $x = \infty$, for which we have $\sigma_\infty = I$ and $S_\infty = B$. Therefore, by the definition of ϑ, $\vartheta^3(S_\infty) = e(1/24)^3 = e(1/8)$, and in particular, $\kappa_\infty = 1/8$. It follows from (6.10) that

$$\theta_{3/2}(z) = \theta_{3/2}{}_{|\sigma_\infty}(z) = e(z/8) \sum_{n=0}^{\infty} \widehat{\theta_{3/2}}_\infty(n) e(nz)$$

where the complex numbers $\widehat{\theta_{3/2}}_{\infty}(n)$ are the Fourier coefficients of $\theta_{3/2}(z)$ at the nonsingular cusp ∞, and

$$h(z) := \sum_{n=0}^{\infty} \widehat{\theta_{3/2}}_{\infty}(n) e(nz)$$

is the Fourier expansion of $\theta_{3/2}(z)$ at ∞. This yields the expression

$$\theta_{\frac{3}{2}+2\ell}(z) = 8^{\ell}\partial_{3/2}^{\ell}\theta_{3/2}(z) = 8^{\ell}\partial_{3/2}^{\ell}\left(e(z/8)h(z)\right). \tag{6.11}$$

Using (6.1) and (6.4) we can express the differential operator $\partial_{3/2}^{\ell}$ as

$$\partial_{3/2}^{\ell} = \sum_{j=0}^{\ell} \binom{\ell}{j} \frac{\Gamma(\ell+\frac{3}{2})}{\Gamma(j+\frac{3}{2})} \left(\frac{-1}{4\pi y}\right)^{\ell-j} \left(\frac{1}{2\pi i}\right)^{j} \frac{d^j}{dz^j}.$$

By the Leibniz rule,

$$\frac{d^j}{dz^j}(e(z/8)h(z)) = e(z/8)\sum_{m=0}^{j} \binom{j}{m} \left(\frac{\pi i}{4}\right)^{m} h^{(j-m)}(z).$$

Apply these facts to the right hand side of (6.11) to obtain

$$\theta_{\frac{3}{2}+2\ell}(z) = e(z/8)8^{\ell}\sum_{j=0}^{\ell}\sum_{m=0}^{j} \binom{\ell}{j}\binom{j}{m} \frac{\Gamma(\ell+\frac{3}{2})}{\Gamma(j+\frac{3}{2})} \left(\frac{-1}{4\pi y}\right)^{\ell-j} \left(\frac{1}{2\pi i}\right)^{j} \left(\frac{\pi i}{4}\right)^{m} h^{(j-m)}(z),$$

from which it follows that $\theta_{\frac{3}{2}+2\ell}(z)$ has exponential decay in the cusp at ∞ for all $\ell \in \mathbb{Z}_{\geq 0}$. □

7. Proof of Theorem A

Define the function

$$F_k(z) = \operatorname{Im}(z)^{k-\frac{1}{2}} |\theta_{k-\frac{1}{2}}(z)|^2.$$

By Proposition 6.1, $\theta_{k-\frac{1}{2}} \in M^*_{k-\frac{1}{2}}(\Gamma, m)$, where $\Gamma = SL_2(\mathbb{Z})$ and $m = \vartheta^3$ if $k \geq 2$ is even, and $\Gamma = \Gamma_0(2)$ and $m = \lambda$ if $k \geq 1$ is odd. In either case for $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma$ we have

$$\begin{aligned} |\theta_{k-\frac{1}{2}}(Az)|^2 &= |m(A)|^2 \left|(\gamma z+\delta)^{k-\frac{1}{2}}\right|^2 |\theta_{k-\frac{1}{2}}(z)|^2 \\ &= |\gamma z+\delta|^{2k-1} |\theta_{k-\frac{1}{2}}(z)|^2 \end{aligned}$$

and

$$\operatorname{Im}(Az) = \frac{\operatorname{Im}(z)}{|\gamma z+\delta|^2},$$

so that

$$\begin{aligned} F_k(Az) &= \operatorname{Im}(Az)^{k-\frac{1}{2}} |\theta_{k-\frac{1}{2}}(Az)|^2 \\ &= \frac{\operatorname{Im}(z)^{k-\frac{1}{2}}}{|\gamma z+\delta|^{2k-1}} |\gamma z+\delta|^{2k-1} |\theta_{k-\frac{1}{2}}(z)|^2 \\ &= F_k(z). \end{aligned}$$

Hence $F_k(z)$ is Γ-invariant and defined on the modular curve $X = \Gamma\backslash\mathbb{H}$.

Recall that $a = N(\mathfrak{a})$, so

$$z_{\mathfrak{a}}^{(2)} = \frac{b + \sqrt{-D}}{2N(\mathfrak{a})},$$

and

$$\mathrm{Im}(z_{\mathfrak{a}}^{(2)}) = \frac{\sqrt{D}}{2N(\mathfrak{a})}.$$

It follows that

$$F_k(z_{\mathfrak{a}}^{(2)}) = \mathrm{Im}(z_{\mathfrak{a}}^{(2)})^{k-\frac{1}{2}} |\theta_{k-\frac{1}{2}}(z_{\mathfrak{a}}^{(2)})|^2 = \frac{D^{\frac{k}{2}-\frac{1}{4}}}{2^{k-\frac{1}{2}}} \frac{|\theta_{k-\frac{1}{2}}(z_{\mathfrak{a}}^{(2)})|^2}{N(\mathfrak{a})^{k-\frac{1}{2}}},$$

or equivalently,

$$\frac{2^{k-\frac{1}{2}}}{D^{\frac{k}{2}-\frac{1}{4}}} F_k(z_{\mathfrak{a}}^{(2)}) = \frac{|\theta_{k-\frac{1}{2}}(z_{\mathfrak{a}}^{(2)})|^2}{N(\mathfrak{a})^{k-\frac{1}{2}}}. \tag{7.1}$$

Substitute (7.1) in (5.1) and use the calculation

$$c_k \cdot \frac{2^{k-\frac{1}{2}}}{D^{\frac{k}{2}-\frac{1}{4}}} = \frac{\pi^k D^{\frac{k}{2}-\frac{3}{4}}}{2^{k-3}(k-1)!} \cdot \frac{2^{k-\frac{1}{2}}}{D^{\frac{k}{2}-\frac{1}{4}}} = \frac{2^{\frac{5}{2}}\pi^{k-1}}{(k-1)!} \frac{\pi}{\sqrt{D}}$$

and the Dirichlet analytic class number formula

$$L((-D/\cdot)), 1) = \frac{\pi h(-D)}{\sqrt{D}}$$

to obtain

$$\sum_{\psi_k \in \Psi_k} L(\psi_k, k) = \frac{2^{\frac{5}{2}}\pi^{k-1}}{(k-1)!} L((-D/\cdot), 1) \sum_{[\mathfrak{a}] \in \mathrm{CL}(K)} F_k(z_{\mathfrak{a}}^{(2)}), \tag{7.2}$$

or equivalently,

$$\frac{1}{h(-D)} \sum_{\psi_k \in \Psi_k} \frac{L(\psi_k, k)}{L\left(\left(\frac{-D}{\cdot}\right), 1\right)} = \frac{2^{\frac{5}{2}}\pi^{k-1}}{(k-1)!} \frac{1}{h(-D)} \sum_{[\mathfrak{a}] \in \mathrm{CL}(K)} F_k(z_{\mathfrak{a}}^{(2)}). \tag{7.3}$$

From here forward we assume that $k \geq 2$ is *even*, which by Proposition 6.1, part *(ii)*, implies that $\theta_{k-\frac{1}{2}} \in S^*_{k-\frac{1}{2}}(SL_2(\mathbb{Z}), \vartheta^3)$. Thus $F_k \in C^\infty(X, d\mu = \frac{dxdy}{y^2})$ with exponential decay in the cusp at ∞ of $X = SL_2(\mathbb{Z})\backslash\mathbb{H}$, and in particular,

$$F_k \in L^2(X, d\mu) = \{F : X \to \mathbb{C} : \int_X |F(z)|^2 \, d\mu < \infty\}.$$

So, by the spectral decomposition of $L^2(X, d\mu)$ with respect to the hyperbolic Laplacian $\Delta = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$, we obtain the spectral expansion

$$F_k(z) = \sum_{n=0}^{\infty} \langle F_k, u_n \rangle_2 \, u_n(z) + \frac{1}{4\pi} \int_{-\infty}^{\infty} \langle F_k, E(\cdot, \frac{1}{2} + it) \rangle_2 E(z, \frac{1}{2} + it) dt, \tag{7.4}$$

which is convergent in the L^2-norm (see [I1, pg. 112]). Here, $u_0(z) = \mathrm{vol}(X)^{-1/2} = \sqrt{3/\pi}$ is the constant eigenfunction for Δ corresponding to the eigenvalue $\lambda_0 = 0$, $\{u_n(z)\}_{n=1}^{\infty}$ is an orthonormal basis of Maass cusp forms satisfying $\Delta u_n = \lambda_n u_n$ where the eigenvalues λ_n are ordered so that $0 < \lambda_1 \le \lambda_2 \le \cdots$, and $E(z,s)$ is the nonholomorphic Eisenstein series corresponding to the cusp at ∞,

$$E(z,s) = \sum_{\gamma \in \Gamma_\infty \backslash SL_2(\mathbb{Z})} (\mathrm{Im}(\gamma z))^s, \quad z \in \mathbb{H}, \quad \mathrm{Re}(s) > 1.$$

Because $\Delta^a F_k \in C^\infty(X, d\mu)$ with exponential decay at ∞ for each integer $a \ge 1$, the spectral expansion (7.4) converges absolutely and uniformly on compact subsets of $\mathbb{H}$ (see [I1, pg. 112]), and one can use standard spectral methods to establish the following estimates for the coefficients in the spectral expansion,

$$c_{n,k} := \langle F_k, u_n \rangle_2 \ll \lambda_n^{-a}, \quad n \ge 1, \tag{7.5}$$

and

$$c_k(t) := \big\langle F_k, E(\cdot, \frac{1}{2} + it) \big\rangle_2 \ll (\frac{1}{4} + t^2)^{-a}, \tag{7.6}$$

for each integer $a \ge 1$. The implied constants depend on F_k and a.

Sum over CM points in (7.4) to obtain

$$\begin{aligned} \sum_{[\mathfrak{a}] \in \mathrm{CL}(K)} F_k(z_{\mathfrak{a}}^{(2)}) =& h(-D)\sqrt{3/\pi} c_{0,k} + \sum_{n=1}^{\infty} c_{n,k} \sum_{[\mathfrak{a}] \in \mathrm{CL}(K)} u_n(z_{\mathfrak{a}}^{(2)}) \\ &+ \frac{1}{4\pi} \int_{-\infty}^{\infty} c_k(t) \sum_{[\mathfrak{a}] \in \mathrm{CL}(K)} E(z_{\mathfrak{a}}^{(2)}, \frac{1}{2} + it) dt. \end{aligned} \tag{7.7}$$

To estimate the contribution of the discrete spectrum we use the following estimate for the Weyl sums associated to the Maass cusp forms $u_n(z)$, $n \ge 1$ (see [DFI2, pg. 35])

$$\sum_{[\mathfrak{a}] \in \mathrm{CL}(K)} u_n(z_{\mathfrak{a}}^{(2)}) \ll \lambda_n^6 D^{\frac{1}{2} - \frac{1}{28} + \epsilon}. \tag{7.8}$$

The proof of (7.8) relies on Duke's [D] nontrivial bound for the Fourier coefficients of weight 1/2 Maass cusp forms.

To estimate the contribution of the continuous spectrum we use the following estimate for the Weyl sum associated to the Eisenstein series $E(z,s)$,

$$\sum_{[\mathfrak{a}] \in \mathrm{CL}(K)} E(z_{\mathfrak{a}}^{(2)}, s) \ll |s|^2 D^{\frac{1}{2} - \frac{1}{16} + \epsilon}, \quad \text{for} \quad \mathrm{Re}(s) = \frac{1}{2}. \tag{7.9}$$

To prove (7.9) one begins with the formula (see [Z])

$$\zeta(2s) \sum_{[\mathfrak{a}] \in \mathrm{CL}(K)} E(z_{\mathfrak{a}}^{(2)}, s) = (D/4)^{s/2} \zeta(s) L((-D/\cdot), s), \tag{7.10}$$

where $\zeta(s)$ is the Riemann zeta function, and applies the Burgess subconvexity bound ([Bu])

$$|L((-D/\cdot), s)| \ll |s| D^{\frac{1}{4} - \frac{1}{16} + \epsilon}, \quad \text{for} \quad \mathrm{Re}(s) = \frac{1}{2},$$

together with a trivial upper bound for $\zeta(s)$ and a classical lower bound for $\zeta(2s)$ (see [T]) in (7.10).

Apply (7.5), (7.6), (7.8), and (7.9) in (7.7), together with Siegel's theorem

$$h(-D) \gg_\epsilon D^{\frac{1}{2} - \epsilon},$$

to obtain

$$\frac{1}{h(-D)}\sum_{[\mathfrak{a}]\in \mathrm{CL}(K)} F_k(z_{\mathfrak{a}}^{(2)}) = \sqrt{3/\pi}c_{0,k} + O_{k,\delta}(D^{-\delta}) \tag{7.11}$$

for all $\delta < 1/28$. By Siegel's theorem the implied constant in (7.11) is ineffective.

Now,

$$c_{0,k} = \sqrt{3/\pi}\int_X F_k(z)d\mu,$$

so that by definition of $F_k(z)$,

$$\sqrt{3/\pi}c_{0,k} = \int_X F_k(z)d\mu(z) = \int_X |\theta_{k-\frac{1}{2}}(z)|^2 \mathrm{Im}(z)^{k-\frac{1}{2}} d\mu(z) = \langle \theta_{k-\frac{1}{2}}, \theta_{k-\frac{1}{2}}\rangle_{\mathrm{Pet}},$$

where $d\mu(z) = (3/\pi)d\mu$ is the invariant hyperbolic measure of mass 1 on X. Finally, combine (7.3) and (7.11) to obtain

$$\frac{1}{h(-D)}\sum_{\psi_k\in\Psi_k}\frac{L(\psi_k,k)}{L(\left(\frac{-D}{\cdot}\right),1)} = \frac{2^{\frac{5}{2}}\pi^{k-1}}{(k-1)!}\langle \theta_{k-\frac{1}{2}}, \theta_{k-\frac{1}{2}}\rangle_{\mathrm{Pet}} + O_{k,\delta}(D^{-\delta})$$

for all $\delta < 1/28$. This completes the proof.

8. Proof of Corollary B

Let $k \geq 2$ be an *even* integer. First observe that (7.2) is equivalent to

$$\sum_{\psi_k\in\Psi_k} L(\psi_k,k) = \frac{2^{\frac{5}{2}}\pi^k}{(k-1)!}\frac{h(-D)^2}{\sqrt{D}}\frac{1}{h(-D)}\sum_{[\mathfrak{a}]\in \mathrm{CL}(K)} F_k(z_{\mathfrak{a}}^{(2)}). \tag{8.1}$$

As a consequence of (7.11) we have

$$\frac{1}{h(-D)}\sum_{[\mathfrak{a}]\in \mathrm{CL}(K)} F_k(z_{\mathfrak{a}}^{(2)}) = \int_X F_k(z)d\mu(z)(1+o_k(1)) \tag{8.2}$$

as $D\to\infty$, where the term $o_k(1)$ is ineffective. Furthermore, by Siegel's theorem,

$$\frac{h(-D)^2}{\sqrt{D}} \gg_\epsilon D^{\frac{1}{2}-\epsilon}. \tag{8.3}$$

Apply estimates (8.2) and (8.3) in (8.1) to obtain

$$\sum_{\psi_k\in\Psi_k} L(\psi_k,k) = \frac{2^{\frac{5}{2}}\pi^k}{(k-1)!}\frac{h(-D)^2}{\sqrt{D}}\int_X F_k(z)d\mu(z)(1+o_k(1)) \gg_{k,\epsilon} D^{\frac{1}{2}-\epsilon}, \tag{8.4}$$

where the implied constant in (8.4) is ineffective. Note that the estimate (8.4) proves that for D sufficiently large there exists a Hecke character $\psi_k\in\Psi_k$ such that $L(\psi_k,k)>0$.

Let $\psi_{k,u}$ be a unitary Hecke character of K of conductor $\sqrt{-D}\mathcal{O}_K$ satisfying

$$\psi_{k,u}(\alpha\mathcal{O}_K) = \varepsilon(\alpha)\left(\frac{\alpha}{|\alpha|}\right)^{2k-1}, \quad \text{for} \quad (\alpha\mathcal{O}_K, \sqrt{-D}\mathcal{O}_K) = 1, \quad k\in\mathbb{Z}_{\geq 1}.$$

Assume in addition that $\psi_{k,u}$ satisfies (1.2). Let $\Psi_{k,u}$ be the set of unitary Hecke characters of the form $\psi_{k,u}$. This set consists of $h(-D)$ characters, and if $\psi_{k,u}$ is any one of them, $\{\psi_{k,u}\xi :\ \xi \in \mathrm{CL}(K)^\wedge\}$. A straightforward calculation yields

$$L(\psi_k, s+k-\frac{1}{2}) = L(\psi_{k,u}, s),$$

so that

$$L(\psi_k, k) = L(\psi_{k,u}, 1/2).$$

Define the completed L–series

$$\Lambda(\psi_{k,u}, s) := (2\pi)^{-s}(D^2)^{\frac{s}{2}}\Gamma(s+k-\frac{1}{2})L(\psi_{k,u}, s).$$

Then $\Lambda(\psi_{k,u}, s)$ is entire, bounded in vertical strips, and satisfies the functional equation

$$\Lambda(\psi_{k,u}, s) = w(\psi_{k,u})\Lambda(\psi_{k,u}, 1-s),$$

where $w(\psi_{k,u}) = \pm 1$ is the root number of $\psi_{k,u}$ (which is the same as the root number of ψ_k). In the functional equation we used that $\psi_{k,u}$ satisfies (1.2).

Define the Fourier series

$$f_{\psi_{k,u}}(z) = \sum_{n=1}^{\infty} n^{k-\frac{1}{2}}\lambda_{\psi_{k,u}}(n)e(nz),$$

where the Hecke eigenvalues are given by

$$\lambda_{\psi_{k,u}}(n) = \sum_{N(\mathfrak{a})=n} \psi_{k,u}(\mathfrak{a}).$$

By Hecke's converse theorem, $f_{\psi_{k,u}}$ is a normalized, primitive, CM newform of weight $2k$ and level D^2 with trivial nebentypus $\chi_{f\psi_{k,u}} \equiv \mathbf{1}$ (see e.g. [FI, pg. 674]).

Duke, Friedlander, and Iwaniec [DFI1, Theorem 3] established the following deep subconvexity bound for $GL(2)$ automorphic L–functions of normalized cusp newforms of weight $k \geq 2$ and level q with trivial nebentypus.

Theorem 8.1. *Let f be a normalized cusp newform of weight $k \geq 2$ and level q with trivial nebentypus. Then for every integer $j \geq 0$ and complex number s such that* $\mathrm{Re}(s) = 1/2$ *we have*

$$L^{(j)}(f, s) \ll q^{\theta+\epsilon}, \tag{8.5}$$

where $\theta = 47/192$ and the implied constant depends on s, j, and ϵ.

Let $(n, D^2) = 1$, and let

$$a(n,k) = n^{k-\frac{1}{2}}\lambda_{\psi_{k,u}}(n)$$

be the eigenvalue of the Hecke operator T_n for the eigenfunction $f_{\psi_{k,u}}$. Because the eigenvalue $a(n,k)$ is given in terms of a Hecke Grössencharakter, one can show that

$$\sum_{\substack{n \leq L \\ (n,D^2)=1}} a(n,k)^2 \gg (D^2)^{-\epsilon} L$$

(see the remark preceding [DFI1, Corollary 5, pg. 223]). Therefore, hypothesis (15) in [DFI1] is satisfied, and one can take $\theta = 29/120$ in (8.5). It follows from (8.5) with $q = D^2$ that

$$L(\psi_{k,u}, 1/2) \ll_\epsilon D^{\frac{1}{2}-\frac{1}{60}+\epsilon}.$$

A simple estimate now yields

$$\sum_{\psi_{k,u}\in\Psi_{k,u}} L(\psi_{k,u}, 1/2) \ll_\epsilon |\{\psi_{k,u} \in \Psi_{k,u} : \ L(\psi_{k,u}, 1/2) > 0\}| D^{\frac{1}{2}-\frac{1}{60}+\epsilon}. \tag{8.6}$$

Finally, combine estimates (8.4) and (8.6) to obtain

$$|\{\psi_{k,u} \in \Psi_{k,u} : \ L(\psi_{k,u}, 1/2) > 0\}| \gg_{k,\delta} D^\delta$$

for all $\delta < 1/60$. This completes the proof.

References

[Bl] V. Blomer, *Non-vanishing of class group L–functions at the central point*, Ann. Inst. Fourier (Grenoble) **54** (2004), 831–847.

[B] G. Bol, *Invarianten linearer Differentialgleichungen*, Abh. Math. Sem. Univ. Hamburg, **16** (1949), 1–28.

[BFH] D. Bump, S. Friedberg, and J. Hoffstein, *Sums of twisted* GL(3) *automorphic L-functions*, Contributions to automorphic forms, geometry, and number theory, 131–162, Johns Hopkins Univ. Press, Baltimore, MD, 2004.

[Bu] D. Burgess, *On character sums and L–series, I*, Proc. London Math. Soc. **12** (1962), 193–206.

[D] W. Duke, *Hyperbolic distribution problems and half-integral weight Maass forms*, Invent. Math. **92** (1988), 73–90.

[DFI1] W. Duke, J. Friedlander, and H. Iwaniec, *Bounds for automorphic L-functions. II*, Invent. Math. **115** (1994), 219–239.

[DFI2] W. Duke, J. Friedlander, and H. Iwaniec, *Class group L-functions*, Duke Math. J. **79** (1995), 1–56.

[FI] E. Fouvry and H. Iwaniec, *A subconvexity bound for Hecke L–functions*, Ann. Scient. Ec. Norm. Sup. **34** (2001), 669–683.

[I1] H. Iwaniec, *Introduction to the spectral theory of automorphic forms*, Biblioteca de la Revista Matemática Iberoamericana. Revista Matemtica Iberoamericana, Madrid, 1995. xiv+247 pp.

[I2] H. Iwaniec, *Topics in classical automorphic forms*, Graduate Studies in Mathematics, 17. American Mathematical Society, Providence, RI, 1997. xii+259 pp.

[IS] H. Iwaniec and P. Sarnak, *The non-vanishing of central values of automorphic L–functions and Landau-Siegel zeros*, Israel J. Math. **120** (2000), 155–177.

[IK] H. Iwaniec and E. Kowalski, *Analytic number theory*, American Mathematical Society Colloquium Publications, 53. American Mathematical Society, Providence, RI, 2004. xii+615 pp.

[J] U. Jannsen, *Mixed motives and algebraic K-theory. With appendices by S. Bloch and C. Schoen.* Lecture Notes in Mathematics, 1400. Springer-Verlag, Berlin, 1990. xiv+246 pp.

[KMV] E. Kowalski, P. Michel, and J. Vanderkam, *Non-vanishing of high derivatives of automorphic L–functions at the center of the critical strip*, J. Reine Angew. Math. **526** (2000), 1–34.

[LX] C. Liu and L. Xu, *The vanishing order of certain Hecke L–functions of imaginary quadratic fields*, J. Number Theory, **108** (2004), 76–89.

[MaY] R. Masri and T. Yang, *Average values of Hecke L–functions via Galois suborbits of Heegner points*, submitted.

[M] R. Masri, *Quantitative nonvanishing of L–series associated to canonical Hecke characters*, Int. Math. Res. Not., Vol. 2007.

[MV] P. Michel and A. Venkatesh, *Heegner points and nonvanishing of Rankin-Selberg L–functions*, Proceedings of the Gauss-Dirichlet Conference, Clay Mathematics Proceedings, Volume 7, 2007.

[MY] S. D. Miller and T. Yang, *Nonvanishing of the central derivative of canonical Hecke L–functions*, Math. Res. Lett. **7** (2000), 263–277.

[MR] H. Montgomery and D. Rohrlich, *On the L-functions of canonical Hecke characters of imaginary quadratic fields. II*, Duke Math. J. **49** (1982), 937–942.

[N] J. Nekovar, *Kolyvagin's method for Chow groups of Kuga-Sato varieties*, Invent. Math. **107** (1992), 99–125.

[RR] D. Ramakrishnan and J. Rogawski, *Average values of modular L-series via the relative trace formula*, Pure Appl. Math. Q. **1** (2005), 701–735.

[RV1] F. Rodriguez-Villegas, *On the square root of special values of certain L-series*, Invent. Math. **106** (1991), 549–573.

[RV2] F. Rodriguez-Villegas, *Square root formulas for central values of Hecke L-series. II*, Duke Math. J. **72** (1993), 431–440.

[RVY] F. Rodriguez-Villegas and T. Yang, *Central values of Hecke L–functions of CM number fields*, Duke Math. J. **98** (1999), 541–564.

[RVZ] F. Rodriguez-Villegas and D. Zagier, *Square roots of central values of Hecke L-series*, Advances in number theory (Kingston, ON, 1991), 81–99, Oxford Sci. Publ., Oxford Univ. Press, New York, 1993.

[Ro] J. Rogawski, *The multiplicity formula for A-packets*, The zeta functions of Picard modular surfaces, 395–419, Univ. Montral, Montreal, QC, 1992.

[R1] D. Rohrlich, *The nonvanishing of certain Hecke L-functions at the center of the critical strip*, Duke Math. J. **47** (1980), 223–232.

[R2] D. Rohrlich, *On the L-functions of canonical Hecke characters of imaginary quadratic fields*, Duke Math. J. **47** (1980), 547–557.

[S] A. Scholl, *Motives for modular forms*, Invent. Math. **100** (1990), 419–430.

[Sh] G. Shimura, *On periods of modular forms* , Math. Ann. **229** (1977), 211–221.

[T] E. C. Titchmarsh, *The theory of the Riemann zeta-function*, Second edition. Edited and with a preface by D. R. Heath-Brown. The Clarendon Press, Oxford University Press, New York, 1986. x+412 pp.

[W] H. Weber, *Lehrbuch der Algebra*, vol. 3, New York, Chelsea, 1961.

[Y1] T. Yang, *Theta liftings and L–functions of Hecke characters*, J. Reine Angew. Math. **485** (1997), 25–53.

[Y2] T. Yang, *Nonvanishing of central Hecke L-values and rank of certain elliptic curves*, Compositio Math. **117** (1999), 337–359.

[Z] D. Zagier, *Eisenstein series and the Riemann zeta function*, Automorphic forms, representation theory and arithmetic (Bombay, 1979), pp. 275–301, Tata Inst. Fund. Res. Studies in Math., 10, Tata Inst. Fundamental Res., Bombay, 1981.

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